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TECHNICAL NOTE 2431

SKIN FRICTION OF INCOMPRESSIBLE TURBULENT BOUNDARY LAYERS UNDER ADVERSE PRESSURE GRADIENTS

By Fabio R. Goldschmied

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

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SUMMARY

Experimental data for skin friction of turbulent boundary layers under adverse pressure gradients from several sources are presented in graphical form. Data obtained by the momentum-balance method are shown to follow a trend opposite to that of data obtained by hot-wire and heat-transfer methods. The momentum equation and total-pressure-measuring techniques are discussed in relation to skin-friction calculations.

The conclusion is reached that momentum method is of questionable value in regions of adverse gradient and should not be relied upon because the sensitivity to small measuring errors and to deviations from assumed flow conditions in test channels is too great. A new integral energy parameter is introduced and its relation to skin-friction data is demonstrated on the basis of available material. A new momentum thickness which includes the turbulent momentum contribution, is also introduced, and it is shown that this thickness may be as much as $7\frac{1}{2}$ percent lower than the conventional momentum thickness, near the turbulent separation point.

INTRODUCTION

The present knowledge concerning the development of turbulent boundary layers under adverse pressure gradients and approaching separation is far from adequate. In particular, the skin friction under adverse pressure gradients cannot be determined with reasonable accuracy because of the lack of enough reliable experimental and theoretical information. Only through positive boundary-layer-control devices, such as suction slots, suction areas, vortex generators, and so forth, can the design engineer be reasonably assured that the boundary layer will develop in a desired manner.

The present analysis was prepared at the NACA Lewis laboratory to assemble and to review in a critical manner the available information concerning the skin-friction trend of an incompressible turbulent boundary layer in regions of adverse pressure gradients in order to show the conflicting data obtained by different experimental methods and to bring out the important characteristics and parameters affecting skin-friction development.

Because of the absence of a known fundamental relation between the mean velocity gradient and the statistical quantities describing the turbulent field in the general case of shear flow, the theoretical approach to the study of turbulent boundary layers has been based almost exclusively on integral methods; possible solutions have been sought that would satisfy, not the local differential relations as given by the Reynolds equations (reference 1), but only "over-all averages" of momentum (reference 2) and energy (references 3 and 4) across a boundary-layer section.

In general, the averaging across the boundary-layer section may be weighted by multiplying the equation of motion before integration by some arbitrary function chosen to give greatest importance to some particular area of the boundary-layer section, such as the area closest to the wall or the area nearest to the free stream. Such weighted integral equations are developed in references 5 to 7.

These methods provide one or more algebraic relations between wall friction and an equal number of parameters employed to describe the mean velocity profile. Thus, even for cases of zero pressure gradient, a solution for boundary-layer development requires an empirical relation for skin friction.

A sufficient quantity of data available for skin friction of low-speed turbulent boundary layers under zero pressure gradients (flat-plate case) has enabled several authors to develop empirical formulas for skin friction as a function of the Reynolds number based on momentum thickness. These formulas are strictly applicable only to the flat-plate case. German authors generally use the Schultz-Grünow formula (reference 8); British and American authors, the Squire and Young (reference 9) and the Falkner (reference 10) formulas.

When the turbulent boundary layer develops under an adverse pressure gradient, both skin friction and mean velocity profile depend in some manner on pressure gradient. In laminar flow, the pressure gradient enters the analysis through a single dimensionless group called the Pohlhausen λ (reference 11). Doenhoff and Tetervin (reference 12), Buri (reference 13), Garner (reference 14), and Kalikhman (reference 15) have proposed the use of various quantities, all analogous to the Pohlhausen λ, to describe the effect of pressure gradient on the turbulent boundary layer.

The following experimental techniques have been employed for the measurement of turbulent skin friction in an adverse pressure gradient:

(a) Total-head survey: Mean velocity profiles, measured by total-head surveys, permit solving the integrated momentum equation for skin friction. This method is in principle simply the reverse of the theoretical approach discussed previously, whereby a skin-friction law is assumed in order to compute mean profiles and development.

(b) Heat transfer (reference 16): A small heating element is inlaid flush with the test surface. Heat is convected away from this element by the flow in the laminar sublayer next to the wall. The film coefficient may be measured and related by calibration to skin friction.

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(c) Hot-wire anemometer (references 17 and 18): Velocity fluctuations affect, by convective cooling, the electric resistivity of a small heated wire. Turbulent fluctuations in the boundary layer may therefore be measured to provide an evaluation of the Reynolds stress close to the test surface.

This report is primarily concerned with a comparative discussion of these techniques, the data obtained by the various investigators, and the resulting theoretical implications.

DISCUSSION

Influence of Surface Roughness, Free-stream Turbulence and Transition History

It is well known that skin-friction values depend on surface-roughness conditions (reference 19 to 21); consideration must also be given, however, to the influence of free-stream turbulence on skin-friction development for a given chordwise position of the transition from laminar to turbulent flow. The transition history of a turbulent boundary layer has been investigated experimentally (reference 22) and it has been found, in the case of zero pressure gradient, that for a given Reynolds number R_θ, the state of the boundary layer is the same, with respect to mean velocity profile and energy spectrum shape, regardless of the manner in which transition has been caused, provided that a sufficient length of run is allowed for settling. This investigation of transition-history effects has proved to be very important; because heretofore it had not been apparent that the turbulent boundary-layer development is independent of the type of transition, which may occur in a multitude of different ways.

Surface roughness and free-stream turbulence both influence the energy transfer from mean to turbulent flow by increasing the turbulent shear and modifying the mean velocity gradient. An example of the influence of free-stream turbulence on the skin friction of a turbulent boundary layer under adverse pressure gradient, as presented by Tillman in reference 21, is shown in figure 1. Percentage wise, the difference in skin-friction values between runs with and without a turbulence grid is quite appreciable and comparison between results of different authors cannot be properly made unless turbulence conditions are exactly specified by relevant statistical quantities along the flow path, which has very seldom been done in the past. 2192

The difference in mean velocity profiles at the same station for a boundary layer on a flat plate with and without a turbulence grid, as presented by Wieghardt (reference 20) is shown in figure 2. The boundary layer thickness is respectively 80 and 50 millimeters, as given in reference 20. Consequently $d\theta/dx$ may also markedly depend on free-stream turbulence. (Symbols are defined in appendix A.)

Most data have been obtained under low-turbulence conditions and great caution must be used in applications to high-turbulence conditions, such as exist in turbo-machinery.

Skin Friction in Adverse Pressure Gradient by Momentum Method

A number of authors, among them Gruschwitz (reference 23), Tillman (reference 21), Wieghardt (reference 24), and Wieghardt and Tillman (reference 25) have measured the skin friction of a turbulent boundary layer under adverse pressure gradient by the Kármán momentum method.

The Kármán momentum equation may be written:

$$\frac{d\theta}{dx} + \theta \frac{U'}{U} (H+2) = C_{f_0} \tag{1}$$

As shown in appendix B, this equation may be obtained by integrating the Reynolds differential equation through the boundary layer, subject to the assumptions arising from Prandtl's concept of a thin boundary layer as applied to turbulent motion. Thus, the mean pressure is taken to be a function of x alone, and the x-derivatives of both mean or fluctuating quantities are neglected in comparison with the y-derivatives of the same quantities. This second assumption is doubtless valid for mean flow quantities, but for fluctuating quantities justification must be based on analysis of experimental evidence. An analysis of this sort will be presented subsequently.

Total-pressure surveys and wall static-pressure measurements suffice to determine the left-hand side of equation (1), which is then identified with the wall skin-friction coefficient C_{f_0}.

For illustration, curves of the skin-friction parameter $C_{f_0}R_\theta^{1/6}$ against x are plotted in figures 3(a) and 3(b) from data of references 25 and 18, respectively. Another example of the variation of C_{f_0} with x is given in figure 1 from data of reference 21. For the cases of figure 3 which represent different pressure gradients, neither surface conditions nor free-stream turbulence have been specified, but transition has been artificially forced, always at the same position, by means of a wire set in place immediately after a suction slot. Pressure distributions for all cases illustrated can be found in the references noted. Falkner's curve has been shown in figure 3 for convenient comparison with what would be corresponding flat-plate values. The curves all show a considerable increase in the direction of flow, whereas by laminar analogy it was expected that the skin friction would decrease monotonically to zero at separation.

The apparent increase of skin friction in the flow direction, which at first appears to be an established experimental fact, has fascinated many researchers. Doenhoff and Tetervin (reference 12) recognized the contrast and the possibility that the total-pressure measurements were too high because of relatively high turbulent fluctuations, but nevertheless accepted the increasing trend as substantially correct. Garner (reference 14), who rejected the skin-friction data of reference 16, also accepted the increasing skin-friction trend. Gruschwitz (reference 23) and Tillman (reference 21) looked for possible causes of error.

No practical use, however, has been made of these data for application to solutions of the Kármán momentum equation for θ; only flat-plate empirical formulas based on R_θ have been used for this purpose, in cases of pressure gradient, on the assumption that θ does not depend strongly on the precise form of the function $C_{f_0}(x)$. Doenhoff and Tetervin (reference 12) and Kalikhman (reference 15) use the Squire and Young formula (reference 9). Garner (reference 14) uses the Falkner formula (reference 10) while the Schultz-Grunow (reference 8) formula is used by German authors.

Verification that the momentum thickness θ computed from the Kármán equation is insensitive to C_{f_0} may be obtained from the data of Schubauer (reference 18); the momentum thickness θ has been computed with his experimental values of H, by using several sets of values for C_{f_0} of which two are of interest here:

(1) The experimental values of C_{f_0} computed according to the Kármán momentum equation from the faired experimental θ points

(2) Falkner's skin-friction values based only on R_θ and with pressure gradient effects neglected

These two values of C_{f_0} (together with a curve obtained with a hot-wire anemometer to be discussed later) are plotted in figure 4. The difference in the resulting θ values, shown in figure 5, is proportionately quite small, so that it seems reasonable to use flat-plate skin-friction formulas, such as Falkner's when θ is to be computed. In other words, at least in the range of these data, θ is really insensitive to changes of C_{f_0}. (In fig. 5 a set of points of "turbulent momentum thickness" and a set of θ values computed on the basis of hot-wire skin friction are also presented for later discussion.) 2192
On the other hand, when skin-friction values are computed from experimental θ data by using the Kármán momentum equation as an algebraic equation for C_{f_0}, it appears that C_{f_0} is very sensitive to small changes in $d\theta/dx$.

<u>One-parameter correlation for C_{f_0}</u>. - From laminar analogy, several authors have attempted to show a one-parameter correlation for C_{f_0} as obtained by the momentum method. Garner (reference 14) attempted to show that $\frac{7}{6} C_{f_0} R_\theta^{1/6}$ is a universal function of $\theta R_\theta^{1/6} \frac{U'}{U}$ and inferred from some experimental data that $\frac{7}{6} C_{f_0} R_\theta^{1/6}$ is constant and equal to 0.007623 for $\theta R_\theta^{1/6} \frac{U'}{U} \leqq -0.01$. The data of references 25 and 18, plotted in the manner of Garner, are shown in figure 6 and seem clearly to disprove his contention.

Wieghardt and Tillman (reference 25) and Tillman (reference 21) attempted to correlate C_{f_0} with the pressure parameter $\theta \frac{U'}{U}$. Figure 7 shows curves of C_{f_0} against $\theta \frac{U'}{U}$, from data of reference 21. Figure 8 shows curves of skin-friction parameter $C_{f_0} R_\theta^{1/6}$ against a similar pressure parameter $\psi \frac{U'}{U}$ from the data of references 25 and 18; it appears that a typical curve is one with a return loop, so that, for a given value of the pressure parameter, two values of the skin-friction parameter are possible. Furthermore, the points are scattered over a wide region instead of forming a universal curve.

These data indicate most clearly that the skin-friction parameter does not depend on the local values of the pressure parameter in the manner that the skin friction depends on local values of R_θ in the case of zero pressure gradient.

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Two-parameter correlation of C_{f_0}. - For a two-parameter correlation of C_{f_0}, a new second parameter must be selected as none appears in the literature. Surface roughness and free-stream turbulence have an effect on the boundary-layer development because the rate of energy transfer from mean to turbulent flow is modified across the boundary-layer section. It was therefore believed reasonable to select a parameter indicative of the profile of energy transfer, in dimensionless form, such as the right-hand side of the mean energy equation (derived in appendix C). This parameter, denoted by E, is

$$E = \frac{1}{3}\left[\int_0^\delta \frac{\overline{u'v'}}{\overline{u}^2} \frac{\partial}{\partial y}\left(\frac{\overline{u}^3}{U^3}\right) dy + \int_0^\delta \frac{\partial}{\partial x}\left(\frac{\overline{u}^3}{U^3}\right) \times \left(\frac{\overline{u'^2} - \overline{v'^2}}{\overline{u}^2}\right) dy\right] + \frac{U'}{U}\int_0^\delta \left(\frac{\overline{u}^3}{U^3} \frac{\overline{u'^2} - \overline{v'^2}}{\overline{u}^2}\right) dy$$

or

$$E = -\frac{d}{dx} \frac{\left(\frac{1}{2}\rho U^3 \psi_t\right)}{\rho U^3}$$

where the energy deficiency thickness ψ_t is

$$\psi_t = \int_0^\delta \frac{\overline{u}}{U}\left[1 - \frac{\overline{u}^2}{U^2}\left(1 + 2\frac{\overline{u'^2} - \overline{v'^2}}{\overline{u}^2}\right)\right] dy$$

As was expected, the skin-friction parameter $C_{f_0} R_\theta^{1/6}$ did not correlate against local values of the parameter E. If, however, C_{f_0} is assumed to be a function of R_θ, $\frac{U'}{U}\psi$, and E, then a plot of $C_{f_0} R_\theta^{1/6}$ against E for constant values of $\frac{U'}{U}\psi$ might be expected to show some degree of correlation.

Typical results of such an attempt at two-parameter correlation are shown in figure 9. It would appear that no matter how small the pressure parameter is chosen $C_{f_0} R_\theta^{1/6}$ increases monotonically as a function of E, possibly along the straight lines shown in figure 9. An appreciable amount of scatter of the points occurs about these lines, but the scatter is not too large to be attributed to experimental or computational error.

The foregoing approach is of value only insofar as the momentum measurements by total-head surveys are substantially correct and the quantity heretofore denoted C_{f_0} maintains a physical significance as the right-hand side of the momentum equation.

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Hot-Wire and Heat-Transfer Measurements

Experimental methods have been developed that allow evaluation of the turbulent shear stress in the vicinity of the wall in regions of adverse pressure gradient. This turbulent shear is considered to be transmitted across the laminar sublayer to the wall according to the relation $\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial \bar{p}}{\partial x}$, obtained from Reynold's equation and evaluated at the wall. Schubauer and Klebanoff (reference 18) have shown that the skin friction, as determined by an X-type hot-wire set as close as possible to the wall, according to the scheme just described, exhibits a monotonic decline in the region of adverse pressure gradient down to zero at separation.

Ludwieg (reference 16) recently developed a heat-transfer type instrument, which, when calibrated dynamically, is capable of direct measurement of skin-friction magnitude and direction. A small heating element, flush with the wall is cooled by convection, losing heat through the laminar sublayer. The heat-transfer coefficient is evaluated experimentally and related by calibration to the local skin-friction coefficient.

This method of measurement, though requiring careful calibration, would appear capable of providing a direct measure of the wall shearing stress and has the additional important advantage that its use probably involves negligible aerodynamic disturbance of the flow, since the instrument is flush with the wall and the velocity profile is independent of the temperature profile for substantially constant property values of the fluid.

Certain data have been obtained with this instrument by Ludwieg and Tillman (reference 26) that show a monotonically declining skin-friction trend for two cases of adverse pressure gradients. These data cannot be regarded as conclusive since pressure gradients were not specified except as to sign, boundary-layer thicknesses were not given, and the separation points were not indicated.

In figure 10, the data from reference 18 and reference 26 are plotted against R_θ; for comparison, the empirical flat-plate skin-friction formulas of references 8 to 10 are also represented. The hot-wire points lie on a curve which seems to be too high to assure a smooth joining with the flat-plate curves. On the other hand, the data of reference 26 join smoothly with the flat-plate curves.

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PROBLEM OF CONFLICTING SKIN-FRICTION RESULTS

Inasmuch as the momentum, hot-wire, and heat-transfer data discussed in the preceding sections are presented by different authors, such data are not necessarily comparable or applicable to the same phenomenon. In Schubauer's case, however, it is possible to compute the momentum skin friction from the data available and to compare it with the hot-wire results; the data points are plotted in figure 4, and show that a rising momentum-skin-friction trend is concomitant with a decreasing hot-wire skin-friction trend. Such a check could not be made from the Ludwieg-Tillman data because of insufficient information.

This discussion suffices to show that the experimental evidence concerning the skin friction in an adverse gradient is glaringly contradictory. The trend shown by the hot-wire method (and supported by the heat-transfer data) may be correct; the trend shown by the momentum survey method may be correct; or both may be in error. The following paragraphs present a preliminary study of the errors inherent in the current methods of obtaining skin-friction results in an adverse pressure gradient.

Possible inadequacies of momentum theory based on two-dimensional stationary mean flow. - The Kármán momentum equation may not be adequate for adverse-pressure-gradient conditions. What has been computed as skin-friction coefficient may actually be the sum of several integral terms, or in other words, the well-known Prandtl boundary-layer assumptions may not satisfactorily apply. No experimental or theoretical evidence has been presented that the Kármán momentum equation does provide an adequate momentum balance for the turbulent boundary layer under adverse pressure gradient and approaching separation, whereas the hot-wire and heat-transfer data tend to show that it does not.

It is therefore necessary to return to the complete equations of motion for two-dimensional mean flow with three-dimensional turbulence, with the assumption that mean turbulent quantities do not vary in the z-direction. In appendix B, the following integral momentum condition is derived from the equations previously described;

$$\frac{d\theta_t}{dx} + \frac{U'}{U} \theta_t (H_t+2) = C_{f_0} + A + B \tag{2}$$

where

$$\theta_t = \int_0^\delta \frac{\bar{u}}{U}\left[1 - \frac{\bar{u}}{U}\left(1 + \frac{\tau_{yy}}{\rho\bar{u}^2} - \frac{\tau_{xx}}{\rho\bar{u}^2}\right)\right]dy$$

and

$$H_t = \frac{\delta^*}{\theta_t}$$

and

$$A = \int_0^\delta \int_\delta^y \left[2\left(\frac{U'^2}{U^2} + \frac{U''}{U}\right)\frac{\tau_{yx}}{\rho U^2} + 4\,\frac{U'}{U}\,\frac{\partial}{\partial x}\left(\frac{\tau_{yx}}{\rho U^2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\tau_{yx}}{\rho U^2}\right)\right] dy\, dy$$

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$$B = 2\,\frac{U'}{U}\int_0^\delta \int_\delta^y \left[\frac{\partial}{\partial x}\left(\frac{\overline{u}}{U}\,\frac{\overline{v}}{U}\right) - 2\,\frac{\overline{v}}{U}\,\frac{\partial}{\partial x}\left(\frac{\overline{u}}{U}\right)\right] dy\, dy\; +$$

$$\int_0^\delta \int_\delta^y \left[\frac{\overline{u}}{U}\,\frac{\partial^2}{\partial x^2}\left(\frac{\overline{v}}{U}\right) - \frac{\overline{v}}{U}\,\frac{\partial^2}{\partial x^2}\left(\frac{\overline{u}}{U}\right)\right] dy$$

The conventional laminar momentum-thickness expression is modified to include turbulent-momentum contributions, which also means that the shape factor H_t is modified accordingly because δ^* remains unchanged. (The turbulent fluctuations by definition do not contribute to mass flow.)

In isotropic turbulence, where $\tau_{xx} \equiv \tau_{yy}$, and in zero turbulence, where $\tau_{xx} \equiv \tau_{yy} \equiv 0$, $\theta = \theta_t$; however, it is well known that the turbulent field must be strongly nonisotropic in shear flow. Only at the free-stream boundary and at the edge of the laminar sublayer will $\tau_{xx} - \tau_{yy}$ actually be zero.

In figures 11 and 12 the mean and turbulent profiles, from references 18 and 20 are shown. Values of θ and θ_t computed from these data are presented in the following table for data from reference 20:

Turbulence grid	θ	θ_t	$\frac{\theta - \theta_t}{\theta_t} \times 100$ (percent)	R_x	Fig.
With	4.48	4.30	4.18	6.25×10^6	12(a)
Without	5.39	5.22	3.26	6.25×10^6	12(b)

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and values based on reference 18 are presented herein:

x	θ	θ_t	$\frac{\theta - \theta_t}{\theta_t} \times 100$ (percent)	R_θ	Fig.
17.5	0.2318	0.2202	5.27	0.213×10^5	11(a)
20.0	.3302	.3146	4.96	.281	11(b)
21.0	.3984	.3796	4.96	.321	11(c)
22.5	.6308	.5988	5.34	.472	11(d)
23.5	.7228	.6916	4.52	.511	11(e)
24.5	.9468	.9076	4.32	.634	11(f)
25.4	1.1812	1.098	7.57	.766	11(g)

From these tabulations it is apparent that the turbulent momentum thickness θ_t can be over 7.5 percent smaller than θ; the difference between $d\theta_t/dx$ and $d\theta/dx$ may be larger, but it is impossible to estimate how much. In figure 5, θ_t is compared with the experimental values of θ and with values of θ computed from the formula $C_{f_0} R_\theta^{1/6} = 0.006534$ (Falkner's formula), and from hot-wire results.

On the right side of the complete momentum equation there are then three terms, of which the first is the skin-friction coefficient C_{f_0}, the second is the contribution from deviation of the turbulent flow-field from the Prandtl assumptions A, and the third is the contribution from the deviation of the mean flow field from the Prandtl assumptions B. A measurable pressure gradient must exist across the boundary-layer section if terms A, B, or the sum assume an appreciable magnitude. In other words, the transverse pressure gradient can originate from longitudinal gradients of the turbulent flow field, the mean flow field, or both.

According to the Prandtl approach, longitudinal gradients can be neglected in comparison with transverse gradients. Prandtl, however, referred to laminar flow fields; in turbulent regimes, his assumptions can be expected to apply only to the mean flow field, as Liepmann and Laufer (reference 26) point out.

On the basis of direct pressure measurements, Schubauer and Klebanoff (reference 18) state that only "barely detectable" pressure differences were found across the boundary layer at some stations; it therefore seems safe to conclude that no appreciable transverse gradient has been found experimentally. On the other hand, computations of terms A and B should result in agreement with this conclusion; that is, A + B should be negligible in comparison with C_{f_0}.

On the basis of the extensive experimental material presented in reference 18, this computation of terms A and B has been attempted. Because of the difficulties of obtaining first and second x-derivatives graphically, however, the accuracy of the results is not sufficient to allow their presentation; A and B are probably indeed negligible up 2192 to the vicinity of the separation point. If A and B are negligible, the right side of the momentum equation is represented by C_{f_0} alone, at least for engineering purposes, and the discrepancies between momentum skin-friction data and hot-wire and heat-transfer skin-friction cannot be attributed to inadequate momentum theory for two-dimensional flow.

As was mentioned in the preceeding section, hot-wire measurement of wall friction involves, according to theoretical considerations, a straight-line extrapolation from a point on the experimentally determined Reynolds stress profile. This point is taken as the point of tangency of the Reynolds stress profile to the straight line of slope $\frac{\partial \tau_{xy}}{\partial y} = \frac{dp}{dx}$. Thus it is assumed that down to the immediate vicinity of the wall, the Reynolds shear is much larger than the molecular shear, and that in this vicinity the curvature of the total shear profile is negligible. The accuracy of these assumptions remains to be evaluated experimentally.

Inadequacy of experimental technique: Velocity measurements in a turbulent field can be made either by a total-head tube or by a hot-wire anemometer. Liepmann and Laufer (reference 27) show that the total-pressure-probe value is consistently higher than the hot-wire value by an appreciable amount. If a total-pressure tube is considered as reading the sum of the mean velocity and of the root-mean-square fluctuating velocity, the effect of this error on the determination of θ will depend, among other things, on the relative fluctuation profile; that is, on the profile of $u'/\bar{u}$. It is well known that $u'/\bar{u}$ can assume values up to 0.40 in the vicinity of the wall. Inasmuch as $d\theta/dx$ is wanted rather than θ itself, it is not clear what contribution to a possible C_{f_0} error is given by high total-pressure readings.

Possibility of secondary flows or violent unsteadiness: As the separation region is approached, secondary flows may be induced in the boundary layer, which invalidates the two-dimensional assumptions. Tillman made an investigation of this effect in rectangular test channels for his doctoral thesis (Göttingen, 1947) and states that an appraisal of the effect of secondary flows gives a C_{f_0} value lower by some 40 percent, as discussed in reference 26. There is a possibility that the turbulent fluctuations become large near separation; such large fluctuations might impart a violently unsteady character to the flow and cause large cross flows. The decidedly nonlinear behavior of turbulent flow is evidenced by the recent observations (reference 22), that turbulent fluctuations occur actually in gusts or bursts in turbulent shear flows with a free boundary. In this event, none of the methods discussed would be adequate in the vicinity of separation.

This discussion of the methods for determining skin friction of turbulent boundary layer under adverse pressure gradients and approaching separation thus indicates that the momentum method is of questionable value because it is too sensitive to errors in the determination of θ values and because it assumes two-dimensionality of flow, which may be difficult to achieve in test channels. It would therefore seem plausible, on the basis of the evidence available, to consider the skin-friction trends observed by the hot-wire and heat-transfer methods to be more reliable than the momentum trend. The evidence should not, however, be regarded as conclusive.

RELATION OF SKIN FRICTION TO RATIO OF ENERGY QUANTITIES AND TO MEAN VELOCITY-PROFILE PARAMETER

For the data of reference 18, a certain integral-energy ratio shows exactly the same trend as the hot-wire skin friction, which may be coincidental inasmuch as no supporting evidence as yet exists.

The quantity

$$\frac{\int_0^\delta \frac{1}{2} \rho \bar{u}^3 \, dy}{\int_0^x \int_0^\delta \tau_{xy} \frac{\partial \bar{u}}{\partial y} \, dy \, dx} - 1$$

is plotted against the experimental hot-wire C_{f_0} in figure 13.

The similarity of the curves is close over the entire range, so that the points bracket the 45° line quite well; furthermore, this similarity also extends into the favorable-gradient region (not shown in fig. 13). In terms of boundary-layer thicknesses, this similarity indicates that C_{f_0} is proportional to $\left[\frac{\delta}{\psi} - \frac{\delta^*}{\psi} - 2\right]$. Thus, in physical terms, the ratio of the volume integral of the turbulent energy production to the section integral of the kinetic energy appears to be related to the local skin friction as determined by the hot-wire method. From their heat-transfer data, Ludwieg and Tillman (reference 26) suggest a formula for the skin-friction coefficient, which is based on the form parameter H and on the Reynolds number R_θ: 2192

$$C_{f_0} = 0.123 \times 10^{-0.678H} R_\theta^{-0.268}$$

This equation has been applied to the data of Schubauer and Klebanoff (reference 18) with the experimental values for H and R_θ. The results are plotted in figure 13; the Ludwieg-Tillman curve falls below that of the hot-wire points; however, it is apparent that the two curves are similar, since very good correlation is obtained when the Ludwieg-Tillman curve is multiplied by 1.52, which is the ratio of the skin-friction values at the initial point ($x = 17.5$).

CONCLUSIONS

The discussion of the behavior of the turbulent boundary layer in adverse pressure gradient leads to the following conclusions:

Surface roughness and stream turbulence affect the skin friction, and thus should be specified in boundary-layer investigations.

The skin friction determined according to the momentum-survey method cannot be considered a function of a single pressure parameter, as in laminar flow. A turbulent-energy production parameter is proposed as an additional quantity upon which the momentum skin friction may depend. This parameter also appears to be related to the skin friction as determined by the hot-wire method.

The skin friction obtained by the momentum method for adverse pressure gradient shows an increasing downstream trend, whereas data obtained by the hot-wire and heat-transfer methods show a decreasing trend.

In order to establish the correct skin-friction trend, it may be necessary to consider violent fluctuations and secondary flows near separation; it would also seem advisable to investigate thoroughly the accuracy of the accepted method of inferring wall friction from the Reynold's stress profile determined by the hot-wire method. The Ludwieg heat-transfer instrument shows promise as an engineering tool for measuring the true skin friction, and should be fully exploited.

The momentum-survey method for the experimental determination of skin friction in an adverse pressure gradient is of doubtful value because this method is highly sensitive to errors in measurement and because secondary flows (not contemplated in the method) probably exist near separation.

2192 Available experimental information does not permit drawing firm conclusions as to the true skin-friction trend in an adverse gradient. If, however, it is assumed that either the momentum method or the hot-wire method yields the correct trend, the weight of evidence would appear to be in favor of the hot-wire results.

National Advisory Committee for Aeronautics,
Lewis Flight Propulsion Laboratory,
Cleveland, Ohio, March 21, 1950.

APPENDIX A

SYMBOLS

The following symbols are used in this report: 2192

A turbulent contribution to deviation from Prandtl's assumptions as given by equation (B17)

B mean flow contribution to deviation from Prandtl's assumptions, as given by equation (B18)

C_{f_0} coefficient of shear at the wall, $\left(\left(\frac{\tau_{xy}}{\rho U^2}\right)_{y=0}\right)$

E dimensionless energy transfer from mean to turbulent flow, as given by equation (C13)

H mean velocity-profile-shape parameter, (δ^*/θ)

H_t profile-shape parameter, accounting for complete momentum deficiency, (δ^*/θ_t)

p instantaneous static pressure

R_x Reynolds number formed with chordwise distance, (Ux/ν)

R_θ Reynolds number formed with momentum thickness, $(U\theta/\nu)$

U free-stream velocity at edge of boundary layer

U' first derivative of U along chordwise direction, (dU/dx)

U'' second derivative of U along chordwise direction, (d^2U/dx^2)

u instantaneous velocity in x-direction

v instantaneous velocity in y-direction

w instantaneous velocity in z-direction

x distance along streamwise or chordwise direction

y distance along direction perpendicular to wall

z distance along direction perpendicular to x-y plane

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δ turbulent-boundary-layer thickness, extending by definition to point where $\dfrac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}$ is equal to arbitrary small value

δ^* displacement thickness, representative of mass-flow deficiency $\left(\displaystyle\int_0^\delta \left(1 - \frac{\overline{u}}{U}\right) dy\right)$

θ momentum thickness, representative of mean-flow momentum deficiency, $\left(\displaystyle\int_0^\delta \frac{\overline{u}}{U}\left(1 - \frac{\overline{u}}{U}\right) dy\right)$

θ_t turbulent momentum thickness, representative of total momentum deficiency, as given by equation (B16)

λ Pohlhausen parameter, $\left(\dfrac{\delta^2 U'}{\nu}\right)$

μ absolute viscosity

ν kinematic viscosity

ρ mass density

τ_{xx} total normal stress acting along x and on y-z plane $\left(\mu\dfrac{\partial\overline{u}}{\partial x} + (-\rho\overline{u'^2})\right)$

τ_{xy} total shear acting along x and on x-z plane, $\left(\mu\dfrac{\partial\overline{u}}{\partial y} + (-\rho\overline{u'v'})\right)$

τ_{yx} total shear acting along y and on y-z plane, $\left(\mu\dfrac{\partial\overline{v}}{\partial x} + (-\rho\overline{u'v'})\right)$

τ_{yy} total normal stress acting along y and on x-z plane $\left(-\mu\dfrac{\partial\overline{u}}{\partial x} + (-\rho\overline{u'^2})\right)$

ψ energy thickness, representative of mean flow-energy deficiency $\left(\int_0^d \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}^2}{U^2}\right)dy\right)$

ψ_t turbulent energy thickness, representative of total energy deficiency, as given by equation (C12)

Primes on small letters indicate fluctuating quantities

Bars above symbols indicate temporal mean quantities

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APPENDIX B

DERIVATION OF INTEGRAL MOMENTUM EQUATION

The Reynolds equations of motion of an incompressible viscous fluid for two-dimensional steady flow with three-dimensional turbulence are

$$\left.\begin{aligned}
\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}&=\nu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}\right)-\frac{\partial}{\partial x}\overline{u'^2}-\frac{\partial}{\partial y}\overline{u'v'}-\frac{\partial}{\partial z}\overline{u'w'}\\
\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y}&=\nu\left(\frac{\partial^2\bar{v}}{\partial x^2}+\frac{\partial^2\bar{v}}{\partial y^2}\right)-\frac{\partial}{\partial x}\overline{u'v'}-\frac{\partial}{\partial y}\overline{v'^2}-\frac{\partial}{\partial z}\overline{v'w'}\\
0&=\frac{\partial}{\partial x}\overline{u'w'}+\frac{\partial}{\partial y}\overline{v'w'}+\frac{\partial}{\partial z}\overline{w'^2}
\end{aligned}\right\}\quad\text{(B1)}$$

The equations of continuity are

$$\left.\begin{aligned}
\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}&=0\\
\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}&=0
\end{aligned}\right\}\quad\text{(B2)}$$

Assuming no variation of turbulent mean quantities in the z-direction and assembling the stress terms produces

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}=\frac{1}{\rho}\frac{\partial}{\partial x}\left[\mu\frac{\partial\bar{u}}{\partial x}+\left(-\rho\overline{u'^2}\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial y}\left[\mu\frac{\partial\bar{u}}{\partial y}+\left(-\rho\overline{u'v'}\right)\right]\quad\text{(B3a)}$$

$$\begin{aligned}
\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y}&=\frac{1}{\rho}\frac{\partial}{\partial x}\left[\mu\frac{\partial\bar{v}}{\partial x}+\left(-\rho\overline{u'v'}\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial y}\left[\mu\frac{\partial\bar{v}}{\partial y}+\left(-\rho\overline{v'^2}\right)\right]\\
0&=\frac{\partial}{\partial x}\overline{u'w'}+\frac{\partial}{\partial y}\overline{v'w'}
\end{aligned}\quad\text{(B3b)}$$

When the following definitions are made,

$$\mu \frac{\partial \bar{u}}{\partial x} + \left(-\rho \overline{u'^2}\right) \equiv \tau_{xx}$$

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$$\mu \frac{\partial \bar{u}}{\partial y} + \left(-\rho \overline{u'v'}\right) \equiv \tau_{xy}$$

$$\mu \frac{\partial \bar{v}}{\partial x} + \left(-\rho \overline{u'v'}\right) \equiv \tau_{yx}$$

$$\mu \frac{\partial \bar{v}}{\partial y} + \left(-\rho \overline{v'^2}\right) \equiv \tau_{yy}$$

and the third equation of motion is neglected, the following equations are obtained:

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial x} \tau_{xx} + \frac{1}{\rho} \frac{\partial}{\partial y} \tau_{xy} \tag{B4a}$$

$$\bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial x} \tau_{yx} + \frac{1}{\rho} \frac{\partial}{\partial y} \tau_{yy} \tag{B4b}$$

When the following relations are used,

$$\bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y} \bar{u} \cdot \bar{v} + \frac{1}{2} \frac{\partial}{\partial x} \bar{u}^2$$

and

$$\bar{u} \frac{\partial \bar{u}}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} \bar{u}^2$$

and

$$\bar{v} \frac{\partial \bar{v}}{\partial y} = - \bar{v} \frac{\partial \bar{u}}{\partial x} = \bar{v} \frac{\partial \bar{u}}{\partial x} - 2 \bar{v} \frac{\partial \bar{u}}{\partial x}$$

equations (B4) may be written:

$$\frac{\partial}{\partial x} \bar{u}^2 + \frac{\partial}{\partial y} \bar{u}\, \bar{v} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial x} \tau_{xx} + \frac{1}{\rho} \frac{\partial}{\partial y} \tau_{xy} \tag{B5a}$$

$$\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} = \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \tau_{yx} + \frac{1}{\rho} \frac{\partial}{\partial y} \tau_{yy} - \frac{\partial}{\partial x} \bar{u}\, \bar{v} + 2 \bar{v} \frac{\partial \bar{u}}{\partial x} \tag{B5b}$$

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Differentiating equation (B5b) with respect to x yields:

$$\frac{1}{\rho}\frac{\partial^2 \overline{p}}{\partial x \partial y} = \frac{1}{\rho}\frac{\partial^2}{\partial x^2}\tau_{yx} + \frac{1}{\rho}\frac{\partial^2 \tau_{yy}}{\partial x\,\partial y} - \frac{\partial^2}{\partial x^2}\overline{u}\;\overline{v} + 2\frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{u}}{\partial x} + 2\,\overline{v}\frac{\partial^2 \overline{u}}{\partial x^2} \qquad \text{(B6)}$$

Inasmuch as the order of differentiation may be exchanged, equation (B6) may be written:

$$\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}\right) = \frac{1}{\rho}\frac{\partial^2}{\partial x^2}\tau_{yx} + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial x}\tau_{yy}\right) + \overline{v}\frac{\partial^2 \overline{u}}{\partial x^2} - \overline{u}\frac{\partial^2 \overline{v}}{\partial x^2} \qquad \text{(B7)}$$

The potential flow impresses on the boundary layer a certain pressure gradient; therefore $\frac{\partial \overline{p}}{\partial x}$ at $y = \delta$ is given by the potential flow. The manner in which $\left|\frac{\partial \overline{p}}{\partial x}\right|_{y=\delta}$ is modified inside the boundary layer is indicated by equation (B7). Integrating from $y = \delta$ to $y = y$ yields

$$\frac{1}{\rho}\int_{\delta}^{y}\frac{\partial}{\partial y}\left(\frac{\partial \overline{p}}{\partial x}\right)dy = \frac{1}{\rho}\left|\frac{\partial \overline{p}}{\partial x}\right|_{y} - \frac{1}{\rho}\left|\frac{\partial \overline{p}}{\partial x}\right|_{y=\delta}$$

Inasmuch as

$$\left|\frac{\partial \overline{p}}{\partial x}\right|_{y=\delta} = -\frac{\partial}{\partial x}\left(\frac{1}{2}\rho U^2\right) = -\rho U U'$$

Then

$$\frac{1}{\rho}\int_{\delta}^{y}\frac{\partial}{\partial y}\left(\frac{\partial \overline{p}}{\partial x}\right)dy = \frac{1}{\rho}\left|\frac{\partial \overline{p}}{\partial x}\right|_{y} + UU'$$

Therefore

$$\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} = -UU' + \frac{1}{\rho}\int_{\delta}^{y}\frac{\partial^2}{\partial x^2}\tau_{yx}\,dy + \frac{1}{\rho}\frac{\partial}{\partial x}\tau_{yy} + \int_{\delta}^{y}\overline{v}\frac{\partial^2}{\partial x^2}\overline{u}\,dy - \int_{\delta}^{y}\overline{u}\frac{\partial^2}{\partial x^2}\overline{v}\,dy \qquad \text{(B8)}$$

Expressing the velocity quantities as dimensionless ratios over the local free-stream velocity U, dividing by U^2, and simplifying yields

$$2\frac{U'}{U}\frac{\overline{u}^2}{U^2}+\frac{\partial}{\partial x}\frac{\overline{u}^2}{U^2}+\frac{\partial}{\partial y}\frac{\overline{u}}{U}\frac{\overline{v}}{U}+\frac{\frac{\partial\overline{p}}{\partial x}}{\rho U^2}=2\frac{U'}{U}\frac{\tau_{xx}}{\rho U^2}+\frac{\partial}{\partial x}\frac{\tau_{xx}}{\rho U^2}+\frac{\partial}{\partial y}\frac{\tau_{xy}}{\rho U^2} \tag{B9a}$$

$$\begin{aligned}\frac{\frac{\partial\overline{p}}{\partial x}}{\rho U^2}=&-\frac{U'}{U}+2\frac{U'}{U}\frac{\tau_{yy}}{\rho U^2}+\frac{\partial}{\partial x}\frac{\tau_{yy}}{\rho U^2}+\\&\int_\delta^y\left[\left(2\frac{U'^2}{U^2}+2\frac{U''}{U}\right)\frac{\tau_{yx}}{\rho U^2}+4\frac{U'}{U}\frac{\partial}{\partial x}\frac{\tau_{yx}}{\rho U^2}+\frac{\partial^2}{\partial x^2}\frac{\tau_{yx}}{\rho U^2}\right]dy+\\&\int_\delta^y\frac{\overline{v}}{U}\left(\frac{U''}{U}\frac{\overline{u}}{U}+2\frac{U'}{U}\frac{\partial}{\partial x}\frac{\overline{u}}{U}+\frac{\partial^2}{\partial x^2}\frac{\overline{u}}{U}\right)dy+\\&\int_\delta^y\frac{\overline{u}}{U}\left(\frac{U''}{U}\frac{\overline{v}}{U}+2\frac{U'}{U}\frac{\partial}{\partial x}\frac{\overline{v}}{U}+\frac{\partial^2}{\partial x^2}\frac{\overline{v}}{U}\right)dy\end{aligned} \tag{B9b}$$

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Substituting for $\dfrac{\frac{\partial \overline{p}}{\partial x}}{\rho U^2}$ from equation (B9b) into equation (B9a) yields:

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$$
\begin{aligned}
&2\frac{U'}{U}\frac{\overline{u}^2}{U^2} + \frac{\partial}{\partial x}\frac{\overline{u}^2}{U^2} + \\
&\quad \frac{\partial}{\partial y}\frac{\overline{u}}{U}\frac{\overline{v}}{U} - \frac{U'}{U} = 2\frac{U'}{U}\frac{\tau_{xx}}{\rho U^2} + \frac{\partial}{\partial x}\frac{\tau_{xx}}{\rho U^2} + \frac{\partial}{\partial y}\frac{\tau_{xy}}{\rho U^2} - \\
&\qquad 2\frac{U'}{U}\frac{\tau_{yy}}{\rho U^2} - \frac{\partial}{\partial x}\frac{\tau_{yy}}{\rho U^2} + \\
&\qquad \int_y^\delta \left[2\left(\frac{U'^2}{U^2} + \frac{U''}{U}\right)\frac{\tau_{yx}}{\rho U^2} + 4\frac{U'}{U}\frac{\partial}{\partial x}\frac{\tau_{yx}}{\rho U^2} + \frac{\partial^2}{\partial x^2}\frac{\tau_{yx}}{\rho U^2}\right]dy + \\
&\qquad \int_y^\delta \frac{\overline{v}}{U}\left(\frac{U''}{U}\frac{\overline{u}}{U} + 2\frac{U'}{U}\frac{\partial}{\partial x}\frac{\overline{u}}{U} + \frac{\partial^2}{\partial x^2}\frac{\overline{u}}{U}\right)dy + \\
&\qquad \int_y^\delta \frac{\overline{u}}{U}\left(\frac{U''}{U}\frac{\overline{v}}{U} + 2\frac{U'}{U}\frac{\partial}{\partial x}\frac{\overline{v}}{U} + \frac{\partial^2}{\partial x^2}\frac{\overline{v}}{U}\right)dy
\end{aligned}
\tag{B10}
$$

From the continuity requirement the following equations are obtained:

$$
-\frac{\overline{v}}{U} = \frac{U'}{U}\int_0^y \frac{\overline{u}}{U}\,dy + \int_0^y \frac{\partial}{\partial x}\frac{\overline{u}}{U}\,dy
$$

$$
\left|-\frac{\overline{v}}{U}\right|_{y=\delta} = \frac{U'}{U}\int_0^\delta \frac{\overline{u}}{U}\,dy + \frac{d}{dx}\int_0^\delta \frac{\overline{u}}{U}\,dy - \frac{d\delta}{dx}
$$

so that

$$\frac{\partial}{\partial x}\frac{\overline{u}}{U} + \frac{\partial}{\partial y}\frac{\overline{v}}{U} = -\frac{U'}{U}\frac{\overline{u}}{U} \tag{B11}$$

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Noting that

$$\left|\frac{\overline{u}}{U}\frac{\overline{v}}{U}\right|_{y=\delta} = -\frac{U'}{U}\int_0^\delta \frac{\overline{u}}{U}\,dy - \frac{d}{dx}\int_0^\delta \frac{\overline{u}}{U}\,dy + \frac{d\delta}{dx} \tag{B12}$$

$$\left|\frac{\tau_{xy}}{\rho U^2}\right|_{y=\delta} = 0 \tag{B13}$$

and that

$$\int_0^\delta \frac{\partial}{\partial x}\frac{\overline{u}^2}{U^2}\,dy = \frac{d}{dx}\int_0^\delta \frac{\overline{u}^2}{U^2}\,dy - \frac{d\delta}{dx} \tag{B14}$$

and integrating equation (B10) across the boundary layer from $y = 0$ to $y = \delta$ yields

$$\frac{d}{dx}\int_0^\delta \left(\frac{\bar{u}}{U} - \frac{\bar{u}^2}{U^2} + \frac{\tau_{xx}}{\rho U^2} - \frac{\tau_{yy}}{\rho U^2}\right) dy \; +$$

$$\frac{U'}{U}\, 2 \int_0^\delta \left(\frac{\bar{u}}{U} - \frac{\bar{u}^2}{U^2} + \frac{\tau_{xx}}{\rho U^2} - \frac{\tau_{yy}}{\rho U^2}\right) dy \; +$$

$$\frac{U'}{U}\int_0 \left(1 - \frac{\bar{u}}{U}\right) dy = \left|\frac{\tau_{xy}}{\rho U^2}\right|_{y=0} +$$

$$\int_0^\delta \int_\delta^y \left[2\left(\frac{U'^2}{U^2} + \frac{U''}{U}\right)\frac{\tau_{yx}}{\rho U^2} + 4\,\frac{U'}{U}\,\frac{\partial}{\partial x}\,\frac{\tau_{yx}}{\rho U^2} + \frac{\partial^2}{\partial x^2}\,\frac{\tau_{yx}}{\rho U^2}\right] dy\; dy \; +$$

$$2\,\frac{U'}{U}\int_0^\delta \int_\delta^y \left(\frac{\partial}{\partial x}\,\frac{\bar{u}}{U}\,\frac{\bar{v}}{U} - 2\,\frac{\bar{v}}{U}\,\frac{\partial}{\partial x}\,\frac{\bar{u}}{U}\right) dy\; dy \; +$$

$$\int_0^\delta \int_\delta^y \left(\frac{\bar{u}}{U}\,\frac{\partial^2}{\partial x^2}\,\frac{\bar{v}}{U} - \frac{\bar{v}}{U}\,\frac{\partial^2}{\partial x^2}\,\frac{\bar{u}}{U}\right) dy\; dy \qquad \text{(B15)}$$

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The individual terms of the complete integral momentum equation will be considered. The integral $\int_0^\delta \left(\frac{\bar{u}}{U} - \frac{\bar{u}^2}{U^2} + \frac{\tau_{xx}}{\rho U^2} - \frac{\tau_{yy}}{\rho U^2}\right) dy$ corresponds to the momentum deficiency thickness and may be written in the following form:

$$\theta_t = \int_0^\delta \frac{\bar{u}}{U}\left[1 - \frac{\bar{u}}{U}\left(1 + \frac{\tau_{yy}}{\rho\bar{u}^2} - \frac{\tau_{xx}}{\rho\bar{u}^2}\right)\right] dy \tag{B16}$$

If equation (B16) is identical to the well-known laminar momentum thickness $\theta = \int_0^\delta \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}}{U}\right) dy$, except for the factor $\left(1 + \frac{\tau_{yy}}{\rho\bar{u}^2} - \frac{\tau_{xx}}{\rho\bar{u}^2}\right)$, which represents the turbulent-momentum contribution.

The integral $\int_0^\delta \left(1 - \frac{\bar{u}}{U}\right) dy$ corresponds to the mass-flow deficiency and may be termed the displacement thickness δ^*.

The term $\left|\frac{\tau_{xy}}{\rho U^2}\right|_{y=0}$ represents the total shear coefficient at the wall, which may be composed of both laminar and turbulent contributions; however, at the wall itself, the shear is believed to be entirely laminar. This quantity $\left|\frac{\tau_{xy}}{\rho U^2}\right|_{y=0}$ is termed C_{f_0}. The double integral

$$\int_0^\delta \int_\delta^y \left[2\left(\frac{U'^2}{U^2} + \frac{U''}{U}\right)\frac{\tau_{yx}}{\rho U^2} + 4\,\frac{U'}{U}\,\frac{\partial}{\partial x}\,\frac{\tau_{yx}}{\rho U^2} + \frac{\partial^2}{\partial x^2}\,\frac{\tau_{yx}}{\rho U^2}\right] dy\,dy \tag{B17}$$

hereinafter termed A, represents the contribution of the anisotropic low-frequency turbulent field to the deviation from the Prandtl boundary-layer assumptions. If the turbulent field were isotropic, τ_{yx} would be zero and term A also would be zero.

The integrals

$$2\,\frac{U'}{U}\int_0^{\delta}\int_{\delta}^{y}\left(\frac{\partial}{\partial x}\,\frac{\overline{u}}{U}\,\frac{\overline{v}}{U} - 2\,\frac{\overline{v}}{U}\,\frac{\partial}{\partial x}\,\frac{\overline{u}}{U}\right)dy\;dy + \int_0^{\delta}\int_{\delta}^{y}\left(\frac{\overline{u}}{U}\,\frac{\partial^2}{\partial x^2}\,\frac{\overline{v}}{U} - \frac{\overline{v}}{U}\,\frac{\partial^2}{\partial x^2}\,\frac{\overline{u}}{U}\right)dy\;dy \tag{B18}$$

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hereinafter termed B, represent the contribution of the mean flow field to the deviation from the Prandtl boundary-layer assumptions.

In conclusion, the complete integral momentum equation may be written as follows:

$$\frac{d\theta_t}{dx} + \frac{U'}{U}\left(2\theta_t + \delta^*\right) = C_{f_0} + A + B$$

For comparison the Kármán momentum equation is

$$\frac{d\theta}{dx} + \frac{U'}{U}\left(2\theta + \delta^*\right) = C_{f_0}$$

APPENDIX C

DERIVATION OF INTEGRAL ENERGY EQUATION

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If the Navier-Stokes equations are multiplied by the corresponding instantaneous velocities and time averages are taken after the instantaneous-energy quantities are formed, equations describing the average energy balance of a two-dimensional mean flow with three-dimensional turbulence are obtained:

$$\left.\begin{aligned}
u \frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + wu \frac{\partial u}{\partial z} &= - \frac{u}{\rho} \frac{\partial p}{\partial x} + \nu u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
v \frac{\partial v}{\partial t} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y} + vw \frac{\partial v}{\partial z} &= - \frac{v}{\rho} \frac{\partial p}{\partial y} + \nu v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
w \frac{\partial w}{\partial t} + wu \frac{\partial w}{\partial x} + wv \frac{\partial w}{\partial y} + w^2 \frac{\partial w}{\partial z} &= - \frac{w}{\rho} \frac{\partial p}{\partial z} + \nu w \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}\right\} \quad (C1)$$

Equations (C1) can be rewritten as follows:

$$\left.\begin{aligned}
\frac{1}{2} \frac{\partial u^2}{\partial t} + \frac{u}{2} \frac{\partial u^2}{\partial x} + \frac{v}{2} \frac{\partial u^2}{\partial y} + \frac{w}{2} \frac{\partial u^2}{\partial z} &= - \frac{u}{\rho} \frac{\partial p}{\partial x} + \nu u \nabla^2 u \\
\frac{1}{2} \frac{\partial v^2}{\partial t} + \frac{u}{2} \frac{\partial v^2}{\partial x} + \frac{v}{2} \frac{\partial v^2}{\partial y} + \frac{w}{2} \frac{\partial v^2}{\partial z} &= - \frac{v}{\rho} \frac{\partial p}{\partial y} + \nu v \nabla^2 v \\
\frac{1}{2} \frac{\partial w^2}{\partial t} + \frac{u}{2} \frac{\partial w^2}{\partial x} + \frac{v}{2} \frac{\partial w^2}{\partial y} + \frac{w}{2} \frac{\partial w^2}{\partial z} &= - \frac{w}{\rho} \frac{\partial p}{\partial z} + \nu w \nabla^2 w
\end{aligned}\right\} \quad (C2)$$

Adding the three equations yields:

$$\frac{1}{2} \frac{\partial}{\partial t} (u^2+v^2+w^2) + \frac{u}{2} \frac{\partial}{\partial x} (u^2+v^2+w^2) + \frac{v}{2} \frac{\partial}{\partial y} (u^2+v^2+w^2) +$$

$$\frac{w}{2} \frac{\partial}{\partial z} (u^2+v^2+w^2) = - \frac{1}{\rho}\left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z}\right) + \nu \left(u \nabla^2 u + v \nabla^2 v + w \nabla^2 w\right) \quad (C3)$$

The following relations may now be written:

$$u = \overline{u} + u'$$

$$v = \overline{v} + v'$$

$$w = \overline{w} + w'$$

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$$p = \overline{p} + p'$$

and

$$u^2 + v^2 + w^2 = q^2$$

$$u'^2 + v'^2 + w'^2 = q'^2$$

It is then assumed that $\overline{w} = 0$ for two-dimensional mean flow; thus

$$\overline{u}^2 + \overline{v}^2 = \overline{q}^2$$

$$\overline{u'^2} + \overline{v'^2} + \overline{w'^2} = \overline{q'^2}$$

and

$$q^2 = (\overline{u}+u')^2 + (\overline{v}+v')^2 + w'^2$$

$$q^2 = \overline{q}^2 + 2\ (\overline{u}u'+\overline{v}v') + q'^2$$

The left-hand side of the equation (C3) thus becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\overline{q}^2 + 2(\overline{u}u'+\overline{v}v') + q'^2\right] + \frac{\overline{u} + u'}{2}\frac{\partial}{\partial x}\left[\overline{q}^2 + 2(\overline{u}u'+\overline{v}v') + q'^2\right] +$$

$$\frac{v + v'}{2}\frac{\partial}{\partial y}\left[q^2 + 2(\overline{u}u'+\overline{v}v') + q'^2\right] + \frac{w'}{2}\frac{\partial}{\partial z}\left[\overline{q}^2 + 2(\overline{u}u'+\overline{v}v') + q'^2\right]$$

Considering the time derivative, analyzing term by term, and taking time averages yield:

$$\overline{\frac{1}{2}\frac{\partial \overline{q}^2}{\partial t}} = 0$$

because the mean motion is steady, and

$$\frac{\partial \overline{\bar{u}u'}}{\partial t} = \frac{\partial}{\partial t}(\overline{\bar{u}u'}) = 0$$

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because the time average of $\bar{u}u'$ is zero.

Similarly,

$$\frac{\partial}{\partial t}\left(\overline{\bar{v}v'}\right) = 0$$

and

$$\frac{1}{2}\overline{\frac{\partial q'^2}{\partial t}} = \frac{1}{2}\frac{\partial}{\partial t}(\overline{q'^2}) = 0$$

because, by definition of the mean, the averaged turbulent quantity is assumed to be independent of time. Analysis of the terms containing x-derivatives yields

$$\overline{\frac{\bar{u}}{2}\frac{\partial \bar{q}^2}{\partial x}} = \frac{\bar{u}}{2}\frac{\partial \bar{q}^2}{\partial x}$$

$$\bar{u}\,\overline{\frac{\partial \bar{u}\,u'}{\partial x}} = \bar{u}\frac{\partial}{\partial x}(\overline{\bar{u}u'}) = 0$$

because $\overline{u'} = 0$,

$$\bar{u}\,\overline{\frac{\partial \bar{v}\,v'}{\partial x}} = \bar{u}\frac{\partial}{\partial x}(\overline{\bar{v}v'}) = 0$$

because $\overline{v'} = 0$, and

$$\overline{\frac{\bar{u}}{2}\frac{\partial q'^2}{\partial x}} = \frac{\bar{u}}{2}\frac{\partial}{\partial x}(\overline{q'^2})$$

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The quantity $\rho\bar{u}\,\frac{\partial}{\partial x}\left(\frac{\bar{q}^2}{2} + \frac{\overline{q'^2}}{2}\right)$ is the net gain of kinetic-energy transport in the flow direction.
Now:

$$\overline{\frac{u'}{2}\,\frac{\partial\bar{q}^2}{\partial x}} = \frac{\overline{u'}}{2}\,\frac{\partial\bar{q}^2}{\partial x} = 0 \quad \text{because} \quad \overline{u'} = 0 \quad \text{by definition}$$

$$\overline{u'\,\frac{\partial(\bar{u}u')}{\partial x}} = \overline{u'\left(u'\,\frac{\partial\bar{u}}{\partial x} + u\,\frac{\partial\overline{u'}}{\partial x}\right)} = \overline{u'u'}\,\frac{\partial\bar{u}}{\partial x} + \frac{\bar{u}}{2}\,\frac{\partial(\overline{u'u'})}{\partial x}$$

$$\overline{u'\,\frac{\partial(\bar{v}v')}{\partial x}} = \overline{u'\left(v'\,\frac{\partial\bar{v}}{\partial x} + \bar{v}\,\frac{\partial v'}{\partial x}\right)} = \overline{u'v'}\,\frac{\partial\bar{v}}{\partial x} + \bar{v}\left(\frac{\partial(\overline{u'v'})}{\partial x} - \overline{v'\,\frac{\partial u'}{\partial x}}\right)$$

The quantity $\overline{\frac{u'}{2}\,\frac{\partial q'^2}{\partial x}}$ is a turbulent energy convection term and is thus obtained for the x-derivative terms:

$$\bar{u}\,\frac{\partial}{\partial x}\left(\frac{\overline{q^2} + \overline{q'^2}}{2}\right) + \overline{u'u'}\,\frac{\partial\bar{u}}{\partial x} + \overline{u'v'}\,\frac{\partial\bar{v}}{\partial x} + \frac{\bar{u}}{2}\,\frac{\partial(\overline{u'u'})}{\partial x} + v\left[\frac{\partial(\overline{u'v'})}{\partial x} - \overline{v'\,\frac{\partial u'}{\partial x}}\right] +$$

$$\overline{\frac{u'}{2}\,\frac{\partial q'^2}{\partial x}}$$

Analysis of the y-derivative terms yields:

$$\overline{\frac{\bar{v}}{2}\,\frac{\partial\bar{q}^2}{\partial y}} = \frac{\bar{v}}{2}\,\frac{\partial\bar{q}^2}{\partial y}$$

$$\overline{\bar{v}\,\frac{\partial\bar{u}u'}{\partial y}} = \bar{v}\,\frac{\partial}{\partial y}\,(\overline{\bar{u}u'}) = 0$$

because $\overline{u'} = 0$ by definition, and

$$\overline{\bar{v}\,\frac{\partial\bar{v}v'}{\partial y}} = \bar{v}\,\frac{\partial}{\partial y}\,(\overline{\bar{v}v'}) = 0$$

because $\overline{v'} = 0$ by definition

$$\overline{\frac{\bar{v}}{2} \frac{q'^2}{y}} = \frac{\bar{v}}{2} \frac{\partial \overline{q'^2}}{\partial y}$$

The term $\rho\bar{v} \frac{\partial}{\partial y}\left(\frac{\bar{q}^2 + \overline{q'^2}}{2}\right)$ is the net gain of kinetic energy of the mean flow in the y direction:

$$\overline{\frac{v'}{2} \frac{\partial \bar{q}^2}{\partial y}} = \frac{\bar{v}'}{2} \frac{\partial \bar{q}^2}{\partial y} = 0$$

because $\bar{v}' = 0$ by definition.

$$\overline{v' \frac{\partial(\bar{u}u')}{\partial y}} = \overline{v'u'} \frac{\partial \bar{u}}{\partial y} + \overline{v'\bar{u} \frac{\partial u'}{\partial y}} = \overline{u'v'} \frac{\partial \bar{u}}{y} + \bar{u} \left[\frac{\partial(\overline{u'v'})}{\partial y} - \overline{u' \frac{\partial v'}{\partial y}}\right]$$

$$\overline{v' \frac{\partial(\bar{v}v')}{\partial y}} = \overline{v'v'} \frac{\partial \bar{v}}{\partial y} + \frac{\bar{v}}{2} \frac{\partial(\overline{v'v'})}{\partial y}$$

$$\overline{\frac{v'}{2} \frac{\partial q'^2}{\partial y}} = \overline{\frac{v'}{2} \frac{\partial q'^2}{\partial y}}$$

The term $\overline{\frac{v'}{2} \frac{\partial q'^2}{\partial y}}$ is a turbulent energy convection term. Thus, for the y derivative terms, there is obtained:

$$\bar{v} \frac{\partial}{\partial y} \left(\frac{\bar{q}^2+\overline{q'^2}}{2}\right) + \overline{u'v'} \frac{\partial \bar{u}}{\partial y} + \overline{v'v'} \frac{\partial \bar{v}}{\partial y} + \bar{u} \left[\frac{\partial(\overline{u'v'})}{\partial y} - \overline{u' \frac{v'}{y}}\right] +$$

$$\frac{\bar{v}}{2} \frac{\partial(\overline{v'v'})}{\partial y} + \overline{\frac{v'}{2} \frac{\partial q'^2}{\partial y}}$$

Analysis of the terms containing the z derivatives yields:

$$\overline{\frac{w'}{2} \frac{\partial q^2}{\partial z}} = \frac{\bar{w}'}{2} \times 0 = 0$$

because $\bar{w}' = 0$ and $\frac{\partial \bar{q}^2}{\partial z} = 0$,

$$\overline{w' \frac{\partial(\bar{u}u')}{\partial z}} = \overline{w'u'}\,\frac{\partial \bar{u}}{\partial z} + \overline{w'\bar{u}\,\frac{\partial u}{\partial z}} = \bar{u}\left[\frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\,\frac{\partial w'}{\partial z}}\right]$$

because $\frac{\partial \bar{u}}{\partial z} = 0$,

$$\overline{w' \frac{\partial(\bar{v}v')}{\partial z}} = \overline{w'v'}\,\frac{\partial \bar{v}}{\partial z} + \overline{w'\bar{v}\,\frac{\partial v'}{\partial z}} = \bar{v}\left(\frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\,\frac{\partial w'}{\partial z}}\right)$$

because $\frac{\partial \bar{v}}{\partial z} = 0$, and

$$\overline{\frac{w'}{2}\,\frac{\partial q'^2}{\partial z}} = \overline{\frac{w'}{2}\,\frac{\partial q'^2}{\partial z}}$$

Thus, for the z-derivative terms,

$$\bar{u}\left[\frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\,\frac{\partial w'}{\partial z}}\right] + \bar{v}\left[\frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\,\frac{\partial w'}{\partial z}}\right] + \overline{\frac{w'}{2}\,\frac{\partial q'^2}{\partial z}}$$

When the pressure terms in equation (C3) are considered and moved to the left side of the equation,

$$\frac{1}{\rho}\left[(\bar{u}+u')\left(\frac{\partial \bar{p}}{\partial x} + \frac{\partial p'}{\partial x}\right) + (\bar{v}+v')\left(\frac{\partial \bar{p}}{\partial y} + \frac{\partial p'}{\partial y}\right) + w'\,\frac{\partial p'}{\partial z}\right]$$

because $\frac{\partial \bar{p}}{\partial z} = 0$, and

$$\frac{1}{\rho}\left(\bar{u}\,\frac{\partial \bar{p}}{\partial x} + \bar{u}\,\frac{\partial p'}{\partial x} + \bar{v}\,\frac{\partial \bar{p}}{\partial y} + \bar{v}\,\frac{\partial p'}{\partial y} + u'\,\frac{\partial \bar{p}}{\partial x} + v'\,\frac{\partial \bar{p}}{\partial y} + u'\,\frac{\partial p'}{\partial x}\;v'\,\frac{\partial p'}{\partial y} + w'\,\frac{\partial p'}{\partial z}\right)$$

Taking time averages and noting that $\bar{p}' = 0$ yield

$$\frac{1}{\rho}\left(\bar{u}\,\frac{\partial \bar{p}}{\partial x} + \bar{v}\,\frac{\partial \bar{p}}{\partial y} + \overline{u'\,\frac{\partial p'}{\partial x}} + \overline{v'\,\frac{\partial p'}{\partial y}} + \overline{w'\,\frac{\partial p'}{\partial z}}\right)$$

Thus, the complete left side of the equation, including the pressure terms, can be written as follows:

$$\bar{u}\,\frac{\partial}{\partial x}\left(\frac{\bar{q}^2+\overline{q'^2}}{2}\right) + \bar{v}\,\frac{\partial}{\partial y}\left(\frac{\bar{q}^2+\overline{q'^2}}{2}\right) + \frac{\bar{u}}{\rho}\,\frac{\partial\bar{p}}{\partial x} + \frac{\bar{v}}{\rho}\,\frac{\bar{p}}{y} + \qquad \text{(a)}$$

$$\overline{\frac{u'}{2}\,\frac{\partial q'^2}{\partial x}} + \overline{\frac{v'}{2}\,\frac{\partial q'^2}{\partial y}} + \overline{\frac{w'}{2}\,\frac{\partial q'^2}{\partial z}} + \overline{\frac{u'}{\rho}\,\frac{\partial p'}{\partial x}} + \overline{\frac{v'}{\rho}\,\frac{\partial p'}{\partial y}} + \overline{\frac{w'}{\rho}\,\frac{\partial p'}{\partial z}} + \qquad \text{(b)}$$

$$\overline{u'u'}\,\frac{\partial\bar{u}}{\partial x} + \overline{u'v'}\,\frac{\partial\bar{v}}{\partial x} + \overline{u'v'}\,\frac{\partial\bar{u}}{\partial y} + \overline{v'v'}\,\frac{\partial\bar{u}}{\partial y} + \qquad \text{(c)}$$

$$\left.\begin{aligned}&\frac{\bar{u}}{2}\,\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\bar{v}}{2}\,\frac{\partial(\overline{v'v'})}{\partial y} + \bar{v}\left[\frac{\partial(\overline{u'v'})}{\partial x} - \overline{v'\,\frac{\partial u'}{\partial x}}\right] + \bar{u}\left[\frac{\partial(\overline{u'v'})}{\partial y} - \overline{u'\,\frac{\partial v'}{\partial y}}\right] + \\ &\bar{u}\left[\frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\,\frac{\partial w'}{\partial z}}\right] + \bar{v}\left[\frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\,\frac{\partial w'}{\partial z}}\right]\end{aligned}\right\} \qquad \text{(d)}$$

The next problem is to reduce the expressions to a more compact and general form:

The group of terms listed as (d) can be rewritten as follows:

$$\bar{u}\;\frac{1}{2}\left[\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\left(\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}\right] +$$

$$\bar{v}\left[\frac{1}{2}\;\frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right)}\right] =$$

$$\bar{u}\left[\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\,\frac{\partial u'}{\partial x}} - \overline{u'\,\frac{\partial v'}{\partial y}} - \overline{u'\,\frac{\partial w'}{\partial z}}\right] +$$

$$\bar{v}\left[\frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\,\frac{\partial v'}{\partial y}} - \overline{v'\,\frac{\partial u'}{\partial x}} - \overline{v'\,\frac{\partial w'}{\partial z}}\right]$$

because

$$\frac{1}{2}\frac{\partial(\overline{u'u'})}{\partial x} = \overline{u'\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}} - \overline{u'\frac{\partial u'}{\partial x}} = \frac{\partial(\overline{u'u'})}{\partial x} - \overline{u'\frac{\partial u'}{\partial x}}$$

and

$$\frac{1}{2}\frac{\partial(\overline{v'v'})}{\partial y} = \overline{v'\frac{\partial v'}{\partial y}} + \overline{v'\frac{\partial v'}{\partial y}} - \overline{v'\frac{\partial v'}{\partial y}} = \frac{\partial(\overline{v'v'})}{\partial y} - \overline{v'\frac{\partial v'}{\partial y}}$$

Inasmuch as, by continuity, $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$, the following equation is obtained:

$$\bar{u}\left[\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z} - \overline{u'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}\right] +$$

$$\bar{v}\left[\frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{v'w'})}{\partial z} - \overline{v'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}\right] =$$

$$\bar{u}\left[\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z}\right] + \bar{v}\left[\frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{v'w'})}{\partial z}\right] \quad \text{(d)}$$

If the group of terms in (c) is combined with the new group of terms in (d), then

$$\overline{u'u'}\frac{\partial\bar{u}}{\partial x} + \bar{u}\frac{\partial(\overline{u'u'})}{\partial x} + \overline{u'v'}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial(\overline{u'v'})}{\partial x} +$$

$$\overline{u'v'}\frac{\partial\bar{u}}{\partial y} + \bar{u}\frac{\partial(\overline{u'v'})}{\partial y} + \overline{v'v'}\frac{\partial v}{\partial y} + \bar{v}\frac{\partial(\overline{v'v'})}{\partial y} +$$

$$\overline{u'w'}\frac{\partial\bar{u}}{\partial z} + \bar{u}\frac{\partial(\overline{u'w'})}{\partial z} + \qquad \text{since } \overline{u'w'}\frac{\partial\bar{u}}{\partial z} = 0 \text{ because } \frac{\partial\bar{u}}{\partial z} = 0$$

$$\overline{v'w'}\frac{\partial\bar{v}}{\partial z} + \bar{v}\frac{\partial(\overline{v'w'})}{\partial z} \qquad \text{since } \overline{v'w'}\frac{\partial\bar{v}}{\partial z} = 0 \text{ because } \frac{\partial\bar{u}}{\partial z} = 0$$

or adding

$$\frac{\partial}{\partial x}(\bar{u}\,\overline{u'u'} + \bar{v}\,\overline{u'v'}) + \frac{\partial}{\partial y}(\bar{u}\,\overline{u'v'} + \bar{v}\,\overline{v'v'}) + \frac{\partial}{\partial z}(\bar{u}\,\overline{u'w'} + \bar{v}\,\overline{v'w'})$$

The group of terms listed as (b) may be written as follows:

$$\overline{\frac{u'}{2}\frac{\partial q'^2}{\partial x}} + \overline{\frac{u'}{\rho}\frac{\partial p'}{\partial x}} + \overline{\frac{v'}{2}\frac{\partial q'^2}{\partial y}} + \overline{\frac{v'}{\rho}\frac{\partial p'}{\partial y}} + \overline{\frac{w'}{2}\frac{\partial q'^2}{\partial z}} + \overline{\frac{w'}{\rho}\frac{\partial p'}{\partial z}}$$

$$= \overline{u'\frac{\partial}{\partial x}\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)} + \overline{v'\frac{\partial}{\partial y}\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)} + \overline{w'\frac{\partial}{\partial z}\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}.$$

$$= \frac{\partial}{\partial x}\left[\overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial y}\left[\overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial z}\left[\overline{w'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] -$$

$$\overline{\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)\cdot\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}$$

By continuity, $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$; thus terms (b) reduce to:

$$\frac{\partial}{\partial x}\left[\overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial y}\left[\overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial z}\left[\overline{w'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right]$$

The group of terms listed as (a) can be rewritten to yield:

$$\bar{u}\cdot\frac{\partial}{\partial x}\left(\frac{\bar{q}^2+\overline{q'^2}}{2}\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{\bar{q}^2+\overline{q'^2}}{2}\right) + \frac{\bar{u}}{\rho}\frac{\partial\bar{p}}{\partial x} + \frac{\bar{v}}{\rho}\frac{\partial\bar{p}}{\partial y}$$

$$= \bar{u}\frac{\partial}{\partial x}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)$$

$$= \frac{\partial}{\partial x}\left[\bar{u}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right] + \frac{\partial}{\partial y}\left[\bar{v}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right] - \left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y}\right)$$

By continuity, $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$; thus terms (a) reduce to:

$$\frac{\partial}{\partial x}\left[\bar{u}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right] + \frac{\partial}{\partial y}\left[\bar{v}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right]$$

It is now possible to give the complete left side of the energy equation in its final form:

$$\frac{\partial}{\partial x}\left[\bar{u}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right] + \frac{\partial}{\partial y}\left[\bar{v}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right)\right] +$$

$$\frac{\partial}{\partial x}\left[\overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial y}\left[\overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial z}\left[\overline{w'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] +$$

$$\frac{\partial}{\partial x}\left(\bar{u}\ \overline{u'u'} + \bar{v}\ \overline{u'v'}\right) + \frac{\partial}{\partial y}\left(\bar{u}\ \overline{u'v'} + \bar{v}\ \overline{v'v'}\right) + \frac{\partial}{\partial z}\left(\bar{u}\ \overline{u'w'} + \bar{v}\ \overline{v'w'}\right) \qquad \text{(C4)}$$

The terms on the right side of the equation are now considered:

$$\nu\left(u\ \nabla^2 u + v\ \nabla^2 v + w\ \nabla^2 w\right) u\ \nabla^2 u$$

$$= (\bar{u}+u')\left[\frac{\partial^2}{\partial x^2}(\bar{u}+u') + \frac{\partial^2}{\partial y^2}(\bar{u}+u') + \frac{\partial^2}{\partial z^2}(\bar{u}+u')\right]$$

$$= (\bar{u}+u')\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)$$

$$= \bar{u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \bar{u}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right) + u'\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) +$$

$$u'\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)$$

When time averages are taken, the following terms are developed:

$$\bar{u}\left(\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2\bar{u}}{\partial y^2}\right) + \overline{u'\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2}\right)}$$

Similarly, the other terms can be developed:

$$\bar{v}\left(\frac{\partial^2\bar{v}}{\partial x^2} + \frac{\partial^2\bar{v}}{\partial y^2}\right) + \overline{v'\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2}\right)}$$

and

$$\overline{w'\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right)}$$

Thus, the expression on the right side will be

$$\nu\left(\bar{u}\ \nabla^2\bar{u} + \bar{v}\ \nabla^2\bar{v} + \overline{u'\ \nabla^2 u'} + \overline{v'\ \nabla^2 v'} + \overline{w'\ \nabla^2 w'}\right) \qquad (C5)$$

This term represents the dissipation of flow energy into heat. The dissipation mechanism is laminar, as can be seen from the form of the expression.

Writing the three basic equations for a two-dimensional flow with three-dimensional turbulence yields

Continuity equations:

$$\left.\begin{aligned} \frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} &= 0 \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 \end{aligned}\right\} \qquad (C6)$$

Momentum and continuity equations:

$$\left.\begin{aligned}
&\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial y} + \frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial x} = \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\overline{u'u'}\right) - \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - \frac{\partial}{\partial z}\left(\overline{u'w'}\right)\\
&\bar{u}\,\frac{\partial \bar{v}}{\partial x} + \bar{v}\,\frac{\partial \bar{v}}{\partial x} + \frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial y} = \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\overline{u'v'}\right) - \frac{\partial}{\partial y}\left(\overline{v'v'}\right) - \frac{\partial}{\partial z}\left(\overline{v'w'}\right)\\
&\qquad\qquad\frac{\partial}{\partial x}(u'w') + \frac{\partial}{\partial y}(v'w') + \frac{\partial}{\partial z}(w'w') = 0
\end{aligned}\right\} \tag{C7}$$

Energy and continuity equation:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\bar{u}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right) + \overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)} + \bar{u}\,\overline{u'u'} + \bar{v}\,\overline{u'v'}\right] +\\
&\frac{\partial}{\partial y}\left[\bar{v}\left(\frac{\bar{q}^2+\overline{q'^2}}{2} + \frac{\bar{p}}{\rho}\right) + \overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)} + \bar{u}\,\overline{u'v'} + \bar{v}\,\overline{v'v'}\right] +\\
&\frac{\partial}{\partial z}\left[\overline{w'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)} + \bar{u}\,\overline{u'w'} + \bar{v}\,\overline{u'w'}\right]\\
&= \nu\left(\bar{u}\,\nabla^2\bar{u} + \bar{v}\,\nabla^2\bar{v} + \overline{u'\,\nabla^2 u'} + \overline{v'\,\nabla^2 v'} + \overline{w'\,\nabla^2 w'}\right)
\end{aligned} \tag{C8}$$

Because momentum, energy, and continuity conditions must be simultaneously satisfied, the momentum conditions can be inserted in the energy equation and equations of both momentum and energy, (and also continuity) can be obtained.

In the energy equation the substitutions

$$\frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial x} = -\,\bar{u}\,\frac{\partial \bar{u}}{\partial x} - \bar{v}\,\frac{\partial \bar{u}}{\partial y} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\overline{u'u'}\right) - \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - \frac{\partial}{\partial z}\left(\overline{u'w'}\right)$$

and

$$\frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial y} = -\,\bar{u}\,\frac{\partial \bar{v}}{\partial x} - \bar{v}\,\frac{\partial \bar{v}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\overline{u'v'}\right) - \frac{\partial}{\partial y}\left(\overline{v'v'}\right) - \frac{\partial}{\partial z}\left(\overline{v'w'}\right)$$

directly yield

$$\frac{\partial}{\partial x}\left[\overline{u}\ \frac{\overline{q'^2}}{2} + \overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial y}\left[\overline{v}\ \frac{\overline{q'^2}}{2} + \overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] +$$

$$\frac{\partial}{\partial z}\left[\overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \overline{u'u'}\ \frac{\partial \overline{u}}{\partial x} +$$

$$\overline{u'v'}\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y}\right) + \overline{v'v'}\ \frac{\partial \overline{v}}{\partial y} = \nu\left(\overline{u'\ \nabla^2 u'} + \overline{v'\ \nabla^2 v'} + \overline{w'\ \nabla^2 w'}\right)$$

Therefore, it must also hold that

$$\frac{\partial}{\partial x}\left[\overline{u}\left(\frac{\overline{q}^2}{2} + \frac{\overline{p}}{\rho}\right)\right] + \frac{\partial}{\partial y}\left[\overline{v}\left(\frac{\overline{q}^2}{2} + \frac{\overline{p}}{\rho}\right)\right] + \overline{u}\left[\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z}\right] +$$

$$\overline{v}\left[\frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{v'w'})}{\partial z}\right] = \nu\left(\overline{u}\ \nabla^2\overline{u} + \overline{v}\ \nabla^2\overline{v}\right)$$

Either the energy equation and the momentum equations or the two energy and momentum equations can therefore be used.

Continuity, momentum, and energy equations:

$$\frac{\partial}{\partial x}\left[\overline{u}\ \frac{\overline{q'^2}}{2} + \overline{u'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \frac{\partial}{\partial y}\left[\overline{v}\left(\frac{\overline{q'^2}}{2}\right) + \overline{v'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] +$$

$$\frac{\partial}{\partial z}\left[\overline{w'\left(\frac{q'^2}{2} + \frac{p'}{\rho}\right)}\right] + \overline{u'u'}\ \frac{\partial \overline{u}}{\partial x} +$$

$$\overline{u'v'}\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y}\right) + \overline{v'v'}\ \frac{\partial \overline{v}}{\partial y} = \nu\left(\overline{u'\ \nabla^2 u'} + \overline{v'\ \nabla^2 v'} + \overline{w'\ \nabla^2 w'}\right) \tag{C9}$$

and

$$\frac{\partial}{\partial x}\left[\bar{u}\left(\frac{\overline{q}^2}{2}+\frac{\bar{p}}{\rho}\right)\right]+\frac{\partial}{\partial y}\left[\bar{v}\left(\frac{\overline{q}^2}{2}+\frac{\bar{p}}{\rho}\right)\right]+\bar{u}\left[\frac{\partial(\overline{u'u'})}{\partial x}+\frac{\partial(\overline{u'v'})}{\partial y}+\frac{\partial(\overline{u'w'})}{\partial z}\right]+$$

$$\bar{v}\left[\frac{\partial(\overline{u'v'})}{\partial x}+\frac{\partial(\overline{v'v'})}{\partial y}+\frac{\partial(\overline{v'w'})}{\partial z}\right]=\nu\left(\bar{u}\ \nabla^2\bar{u}+\bar{v}\ \nabla^2\bar{v}\right) \tag{C10}$$

For the purposes of this report, equation (C10) is integrated across the boundary-layer section from $y = 0$ to $y = \delta$. This equation of momentum and energy has been chosen because it deals primarily with mean quantities. The viscous dissipation on the right side of equation (C10) can be neglected in comparison with the turbulent-energy production. On the assumptions that $\partial\bar{p}/\partial y$ is actually negligible, that

$$\frac{1}{U^3}\frac{\bar{u}}{\rho}\frac{d\bar{p}}{dx}=-\frac{U'}{U}\frac{\bar{u}}{U}-2\frac{U'}{U}\frac{\overline{v'^2}}{U^2}-\frac{\partial}{\partial x}\left(\frac{\bar{u}}{U}\right)\frac{\overline{v'^2}}{U^2}+\frac{\overline{v'^2}}{U^2}\frac{\partial}{\partial x}\left(\frac{\bar{u}}{U}\right)$$

and that $\bar{u}^2+\bar{v}^2 \equiv \bar{u}^2$, equation (C10) is integrated to the following form:

$$\frac{d}{dx}\int_0^\delta \frac{\bar{u}}{U^2}\left[1-\frac{u^2}{U^2}\left(1+2\,\frac{\overline{u'^2}-\overline{v'^2}}{u^2}\right)\right]dy+3\,\frac{U'}{U}\int_0^\delta \frac{\bar{u}}{U}\left[1-\frac{\bar{u}^2}{U^2}\left(1+2\,\frac{\overline{u'^2}-\overline{v'^2}}{u^2}\right)\right]dy$$

$$=-2\left\{\frac{1}{3}\left[\int_0^\delta \frac{\overline{u'v'}}{u^2}\frac{\partial}{\partial y}\left(\frac{\bar{u}^3}{U^3}\right)dy+\int_0^\delta \frac{\partial}{\partial x}\left(\frac{\bar{u}^3}{U^3}\right)\left(\frac{\overline{u'^2}-\overline{v'^2}}{u^2}\right)dy\right]+\right.$$

$$\left.\frac{U'}{U}\int_0^\delta \frac{\bar{u}^3}{U^3}\left(\frac{\overline{u'^2}-\overline{v'^2}}{u^2}\right)dy\right\} \tag{C11}$$

The right side of the preceding equation is twice the integrated turbulent-energy-production terms from equation (C9); the factor 2 is introduced because on the left side there is twice the kinetic-energy loss.

$$\int_0^\delta \frac{\overline{u}}{U}\left[1 - \frac{\overline{u}^2}{U^2}\left(1 + 2\,\frac{\overline{u'^2} - \overline{v'^2}}{u^2}\right)\right] dy = \psi_t \tag{C12}$$

and

$$\frac{1}{3}\left[\int_0^\delta \frac{\overline{u'v'}}{u^2}\,\frac{\partial}{\partial y}\left(\frac{\overline{u}^3}{U^3}\right) dy + \int \frac{\partial}{\partial x}\,\frac{\overline{u}^3}{U^3}\left(\frac{\overline{u'^2} - \overline{v'^2}}{u^2}\right) dy\right] +$$

$$\frac{U'}{U}\int_0^\delta \frac{\overline{u}^3}{U^3}\left(\frac{\overline{u'^2} - \overline{v'^2}}{u^2}\right) dy = E \tag{C13}$$

give the integral momentum and energy equation as

$$\frac{d\psi_t}{dx} + 3\,\frac{U'}{U}\,\psi_t = -\,2E \tag{C14}$$

or

$$\frac{d}{dx}\,\psi_t\,U^3 = -\,2U^3E$$

Physically, the preceding equation should be written

$$\frac{d}{dx}\left(\frac{1}{2}\,\rho U^3 \psi_t\right) = -\,\rho U^3 E$$

or

$$E = -\,\frac{\frac{d}{dx}\left(\frac{1}{2}\,U^3 \psi_t\right)}{U^3}$$

The loss of kinetic energy $\frac{1}{2}\,\rho U^3 \psi_t$ is equal to the total turbulent-energy production over the boundary-layer volume $\int_0^x \rho U^3 E\,dx$.

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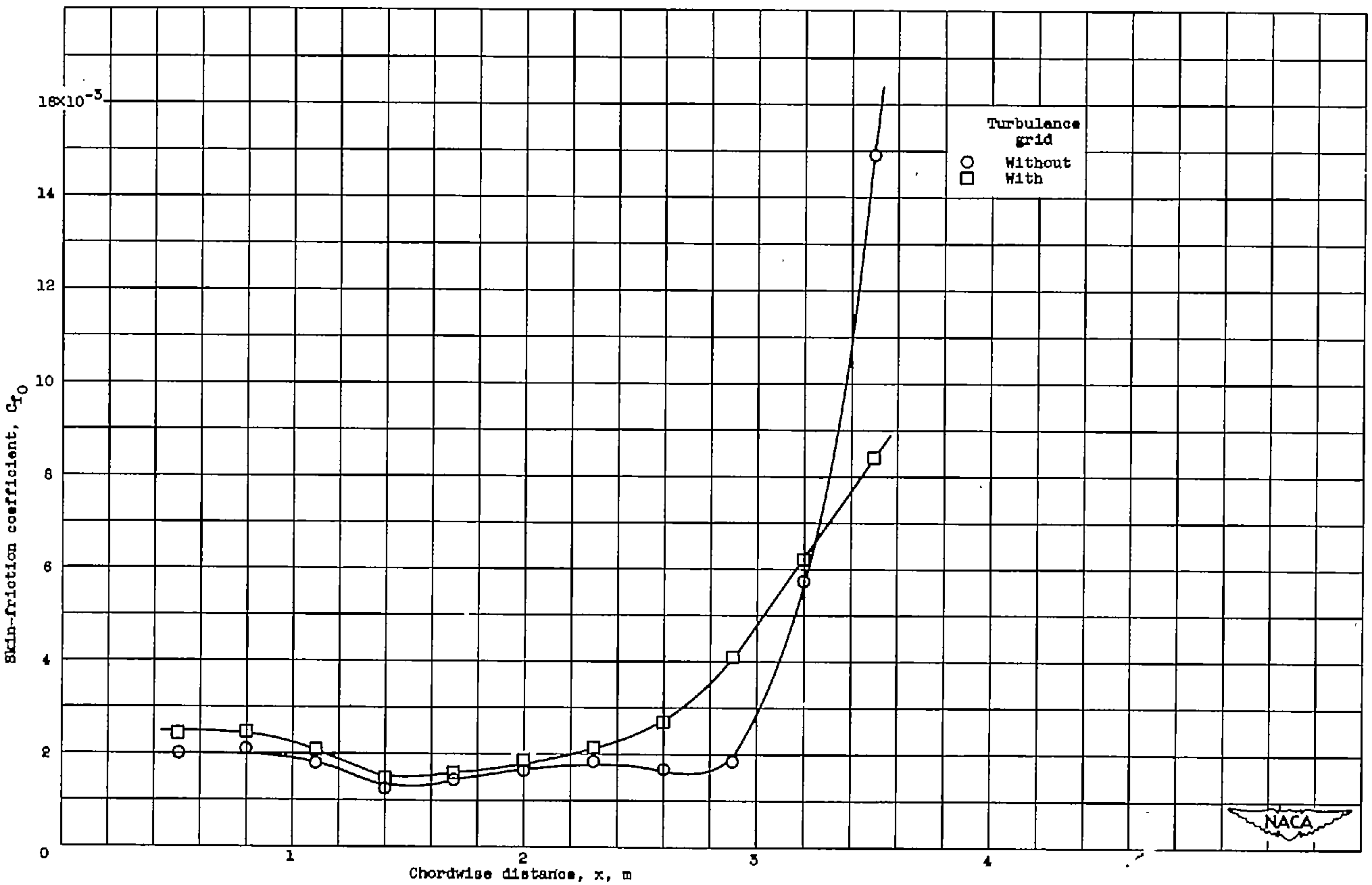


Figure 1. - Skin-friction coefficient for two different free-stream turbulence levels under adverse pressure gradient. Data from reference 21. (Skin friction computed by momentum balance.)

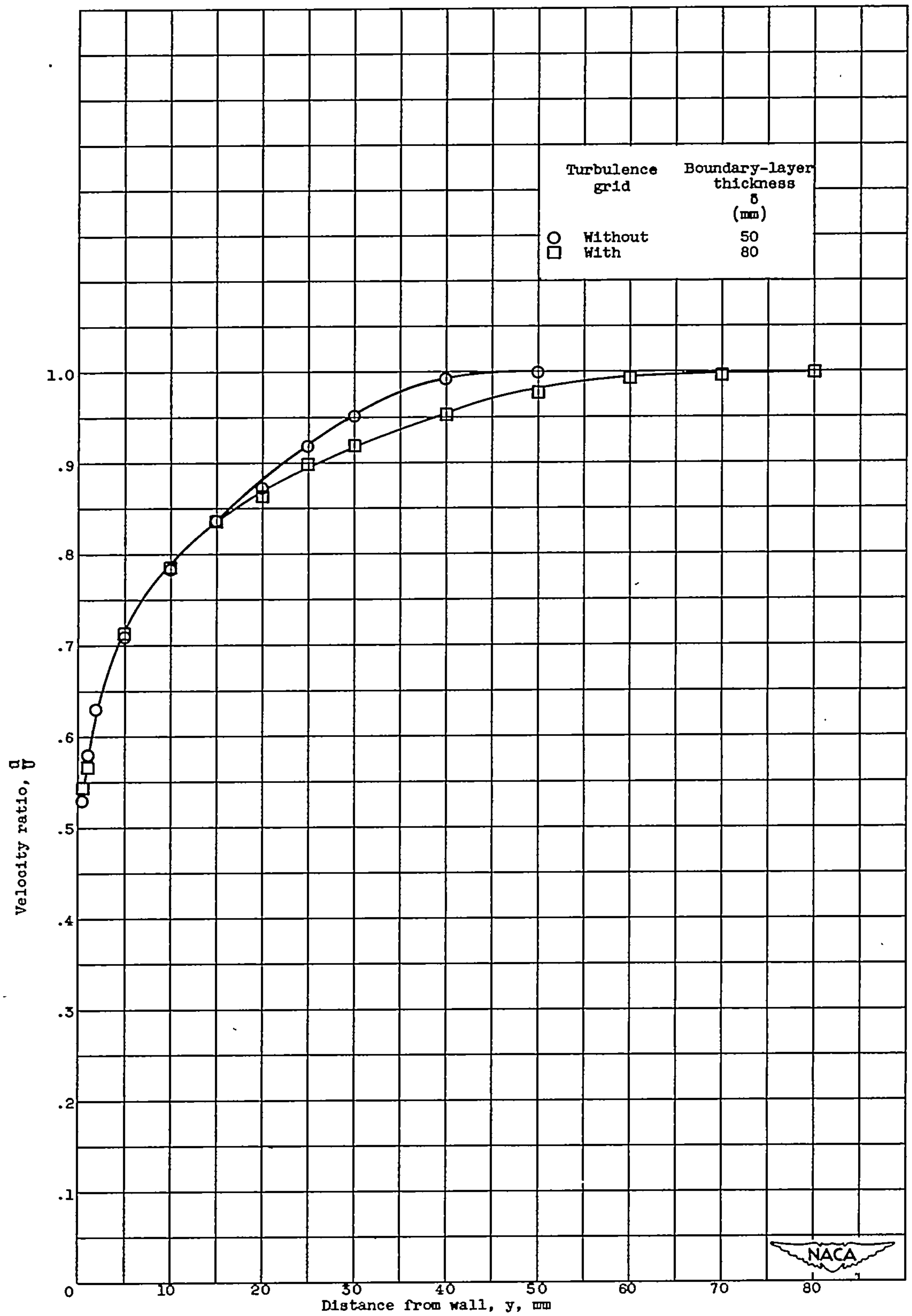
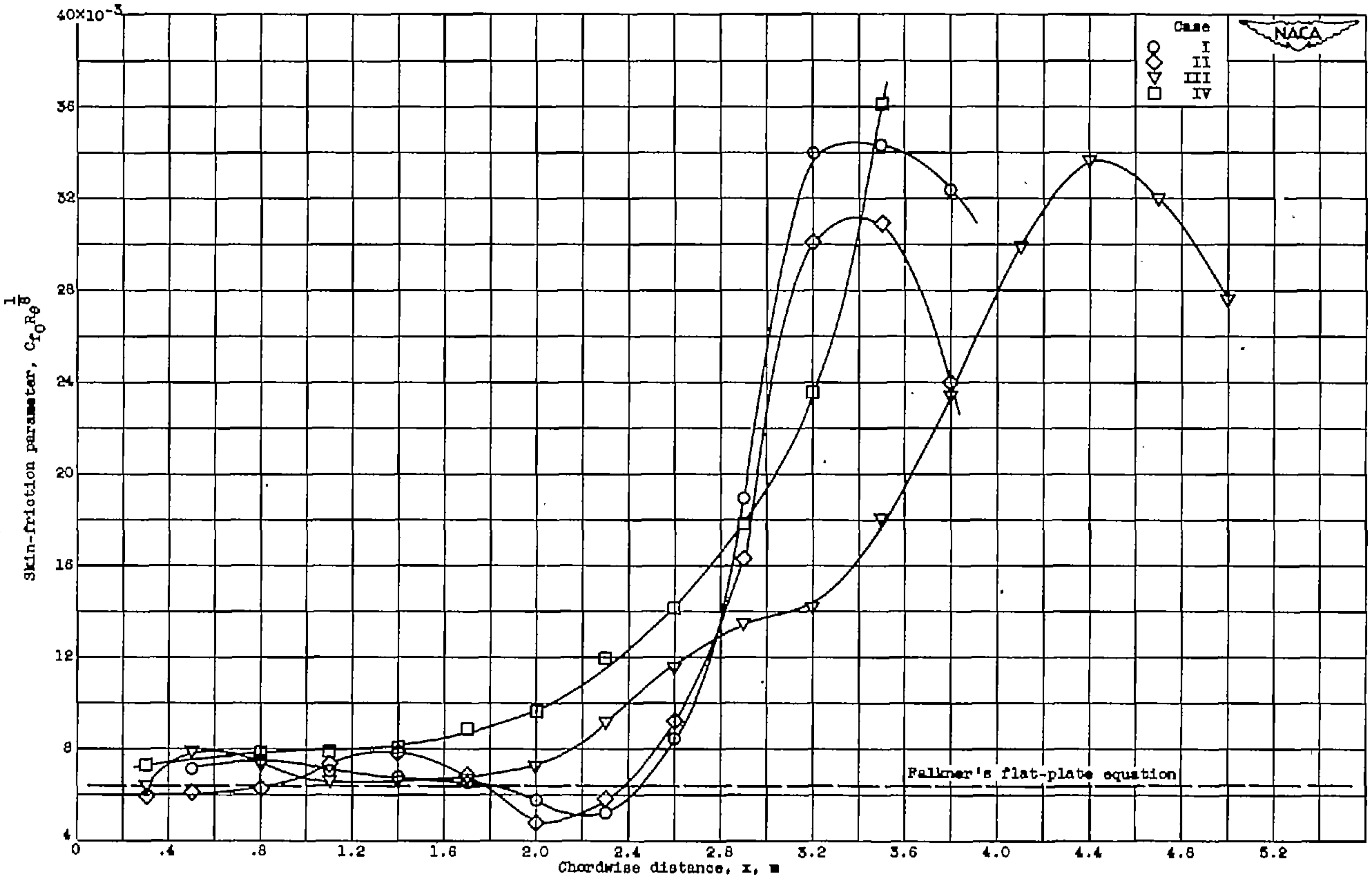
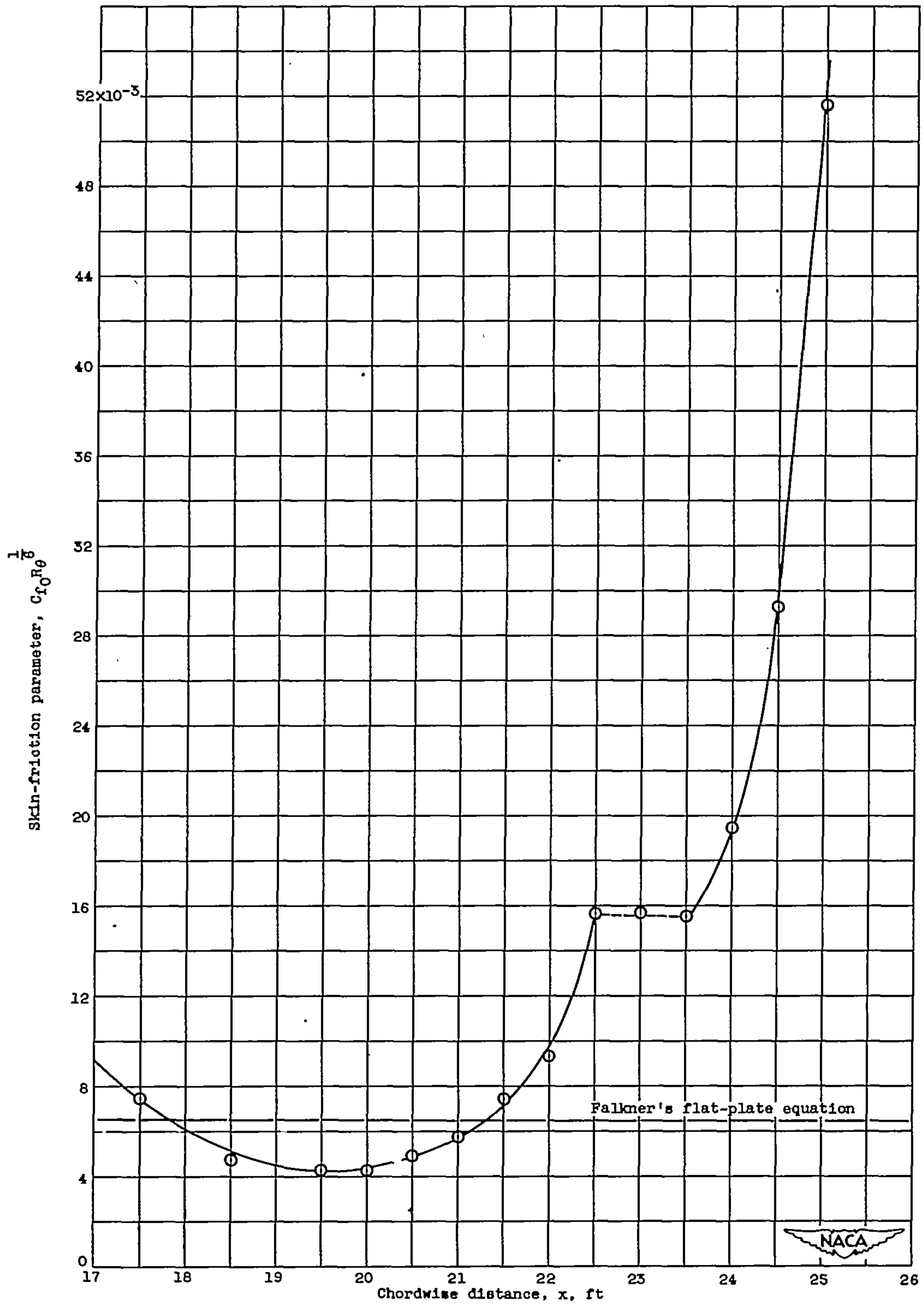


Figure 2. - Mean velocity profiles at same station for various free-stream turbulence levels. Data from reference 20.



(a) Data from reference 25.

Figure 3. - Skin-friction parameter under adverse pressure gradient. (Skin friction computed by momentum method.)



(b) Data from reference 18.

Figure 3. - Concluded. Skin-friction parameter under adverse pressure gradient. (Skin friction computed by momentum method.)

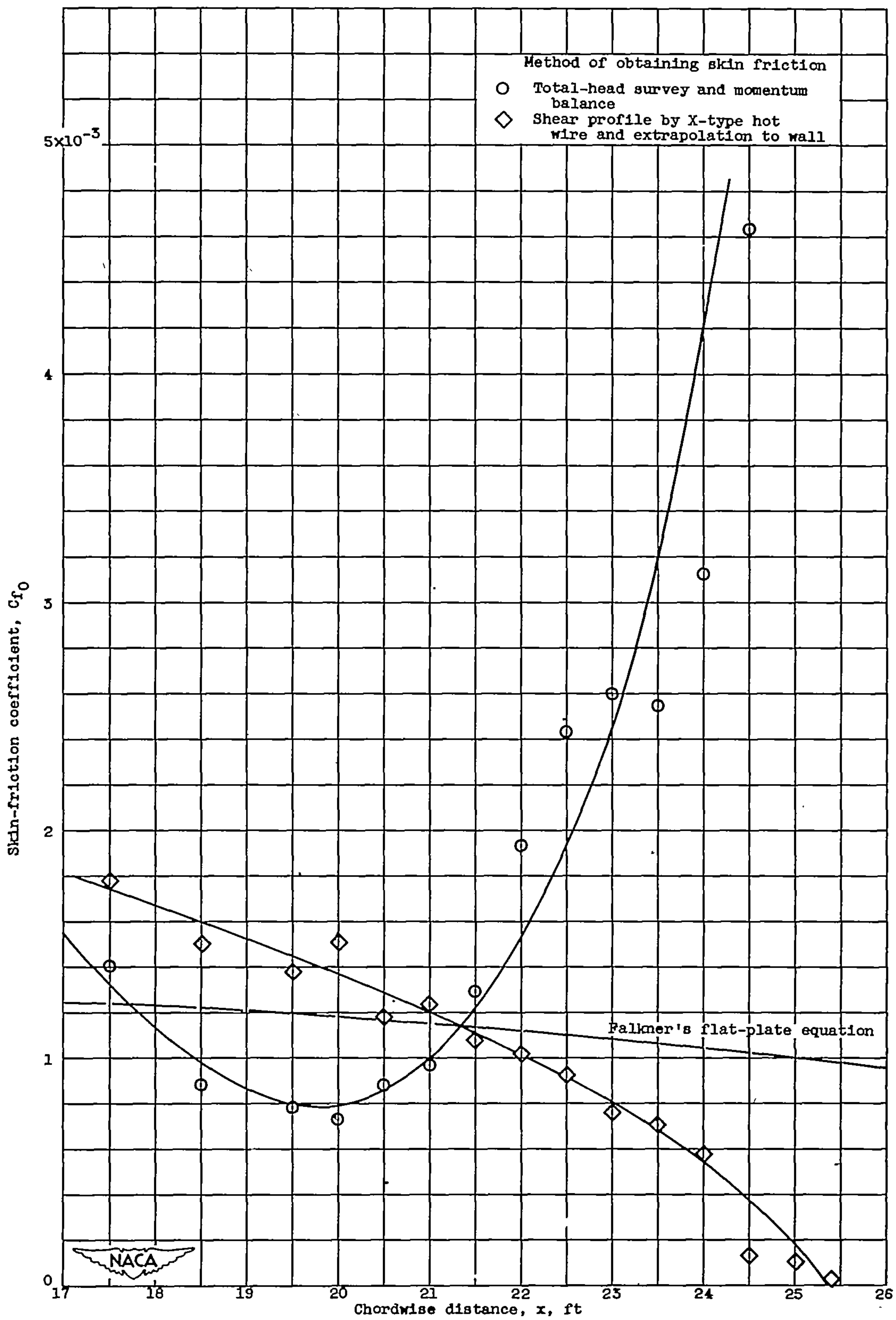


Figure 4. - Comparative momentum and hot-wire skin-friction coefficients under adverse pressure gradient. Data from reference 18.

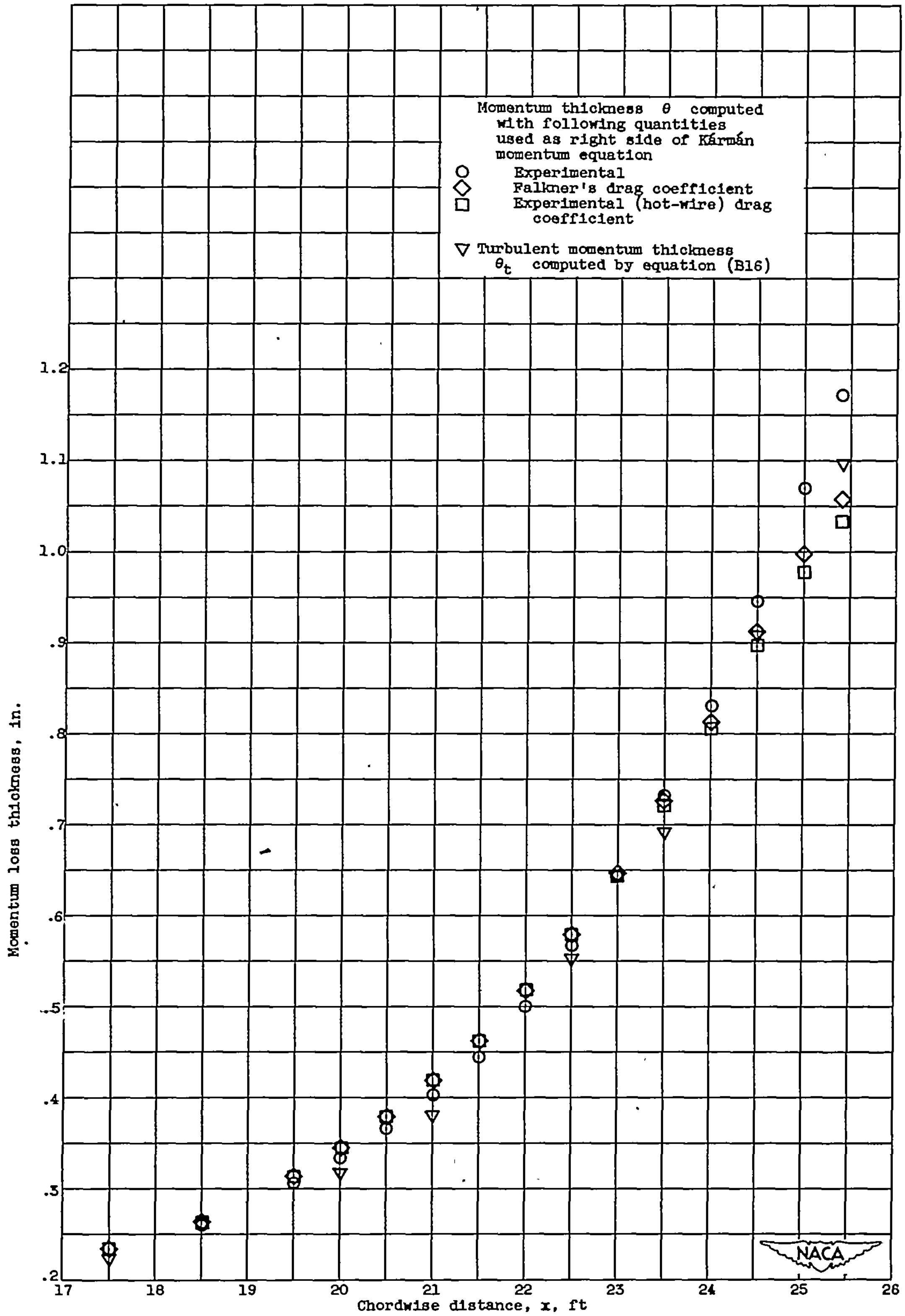


Figure 5. - Momentum thickness under adverse pressure gradient. Data from reference 18.

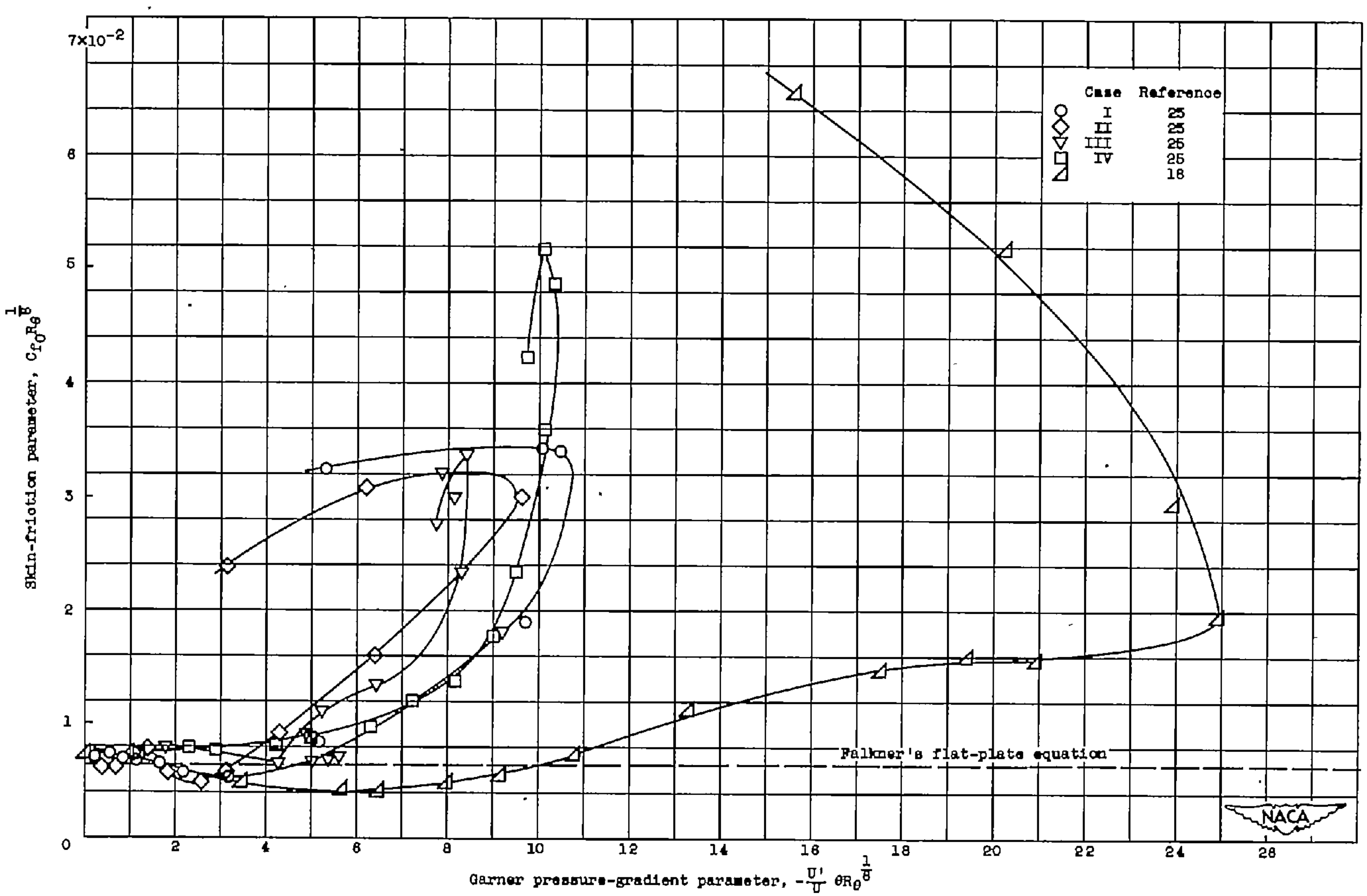


Figure 6. - Relation of skin-friction parameter and Garner's pressure parameter for adverse-pressure-gradient region. Data from references 18 and 25. (Skin friction computed by momentum method.)

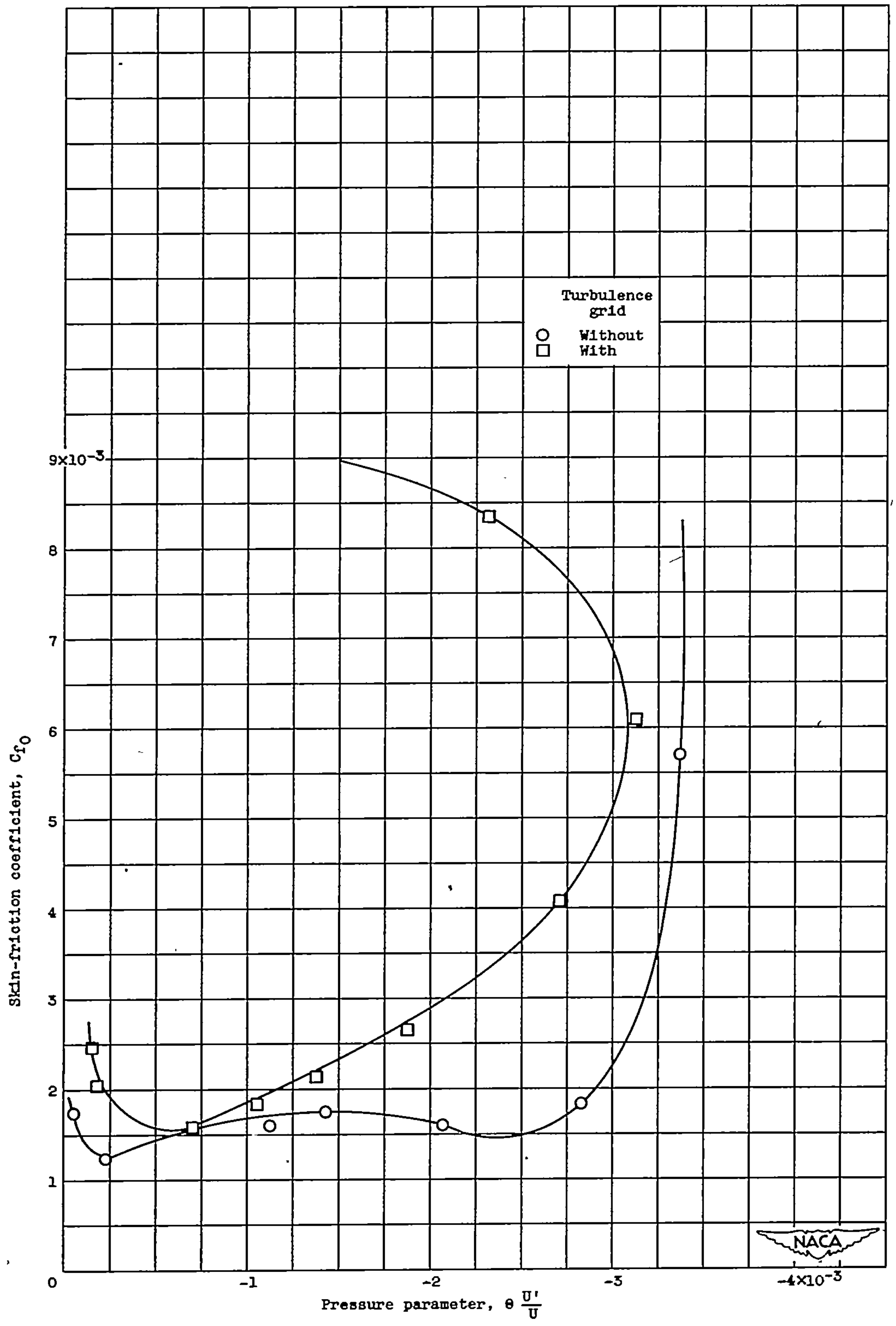


Figure 7. - Relation of skin friction and pressure parameter for two different free-stream turbulence levels under adverse pressure gradients. Data from reference 21. (Skin friction computed by momentum method.)

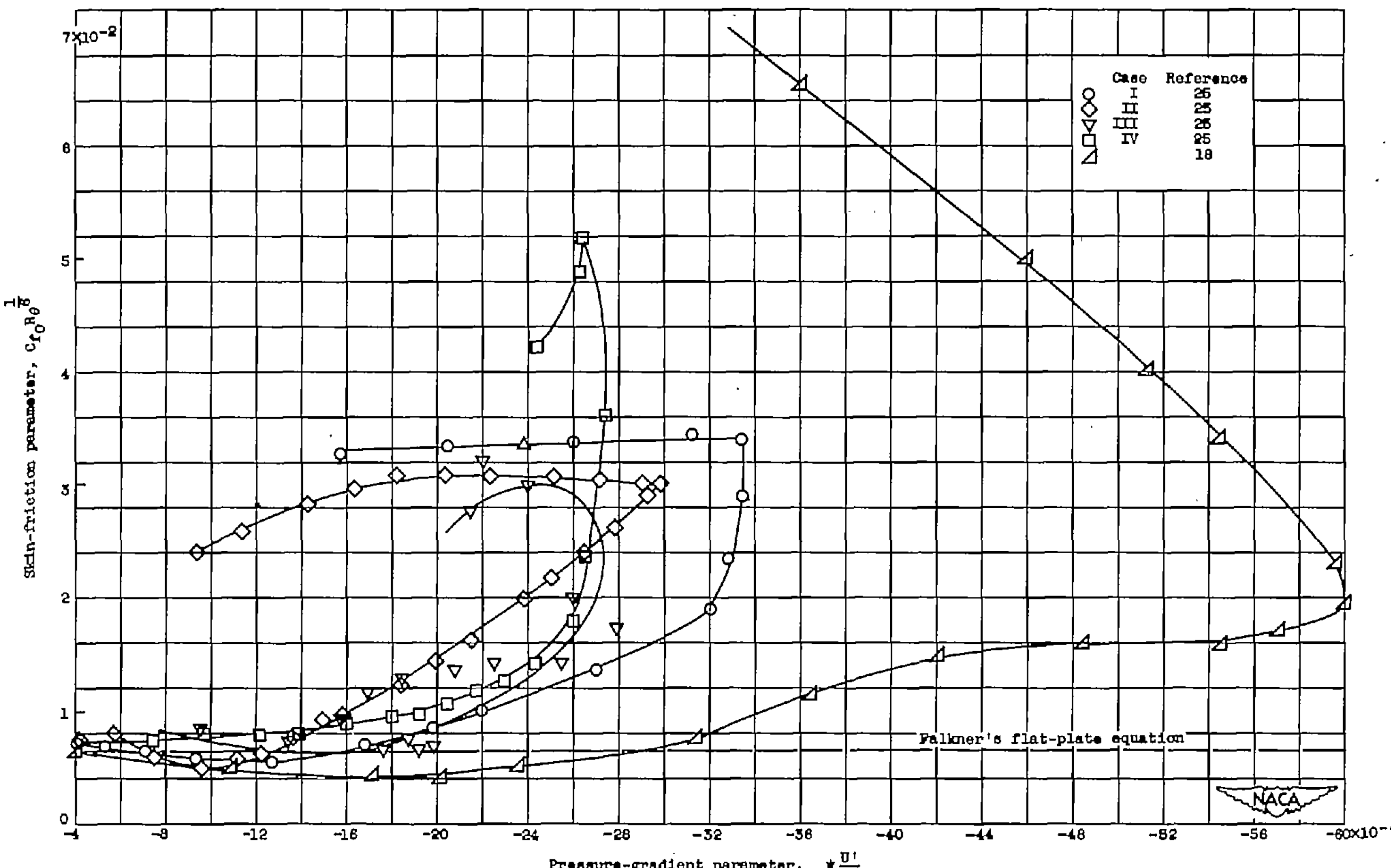


Figure 8. - Relation of skin-friction parameter and pressure-gradient parameter for adverse-pressure-gradient region. Data from references 18 and 25. (Skin friction computed by momentum method.)

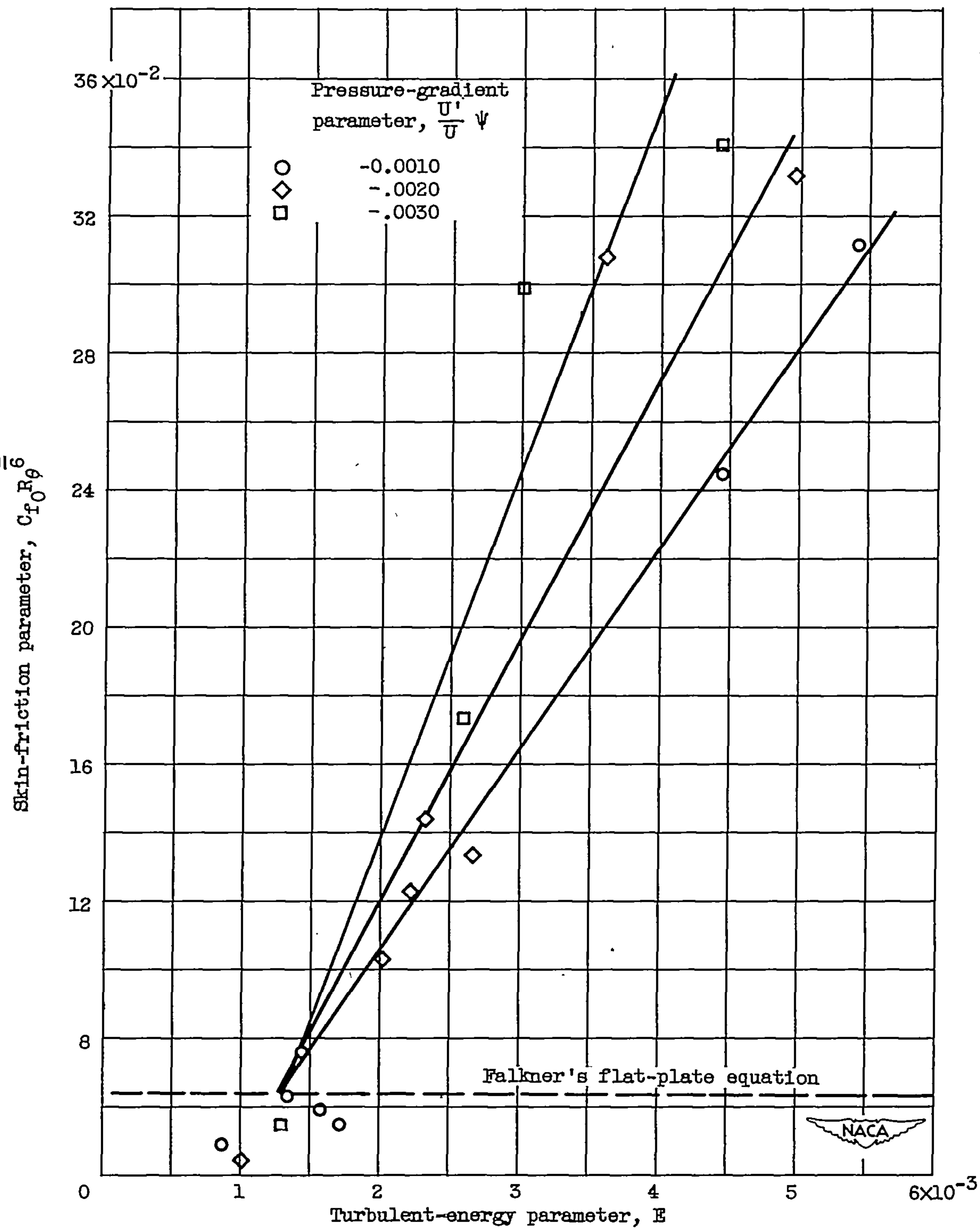


Figure 9. - Relation of skin-friction parameter and turbulent-energy parameter for constant values of pressure parameter. Data from references 18 and 25. (Skin friction computed by momentum method.)

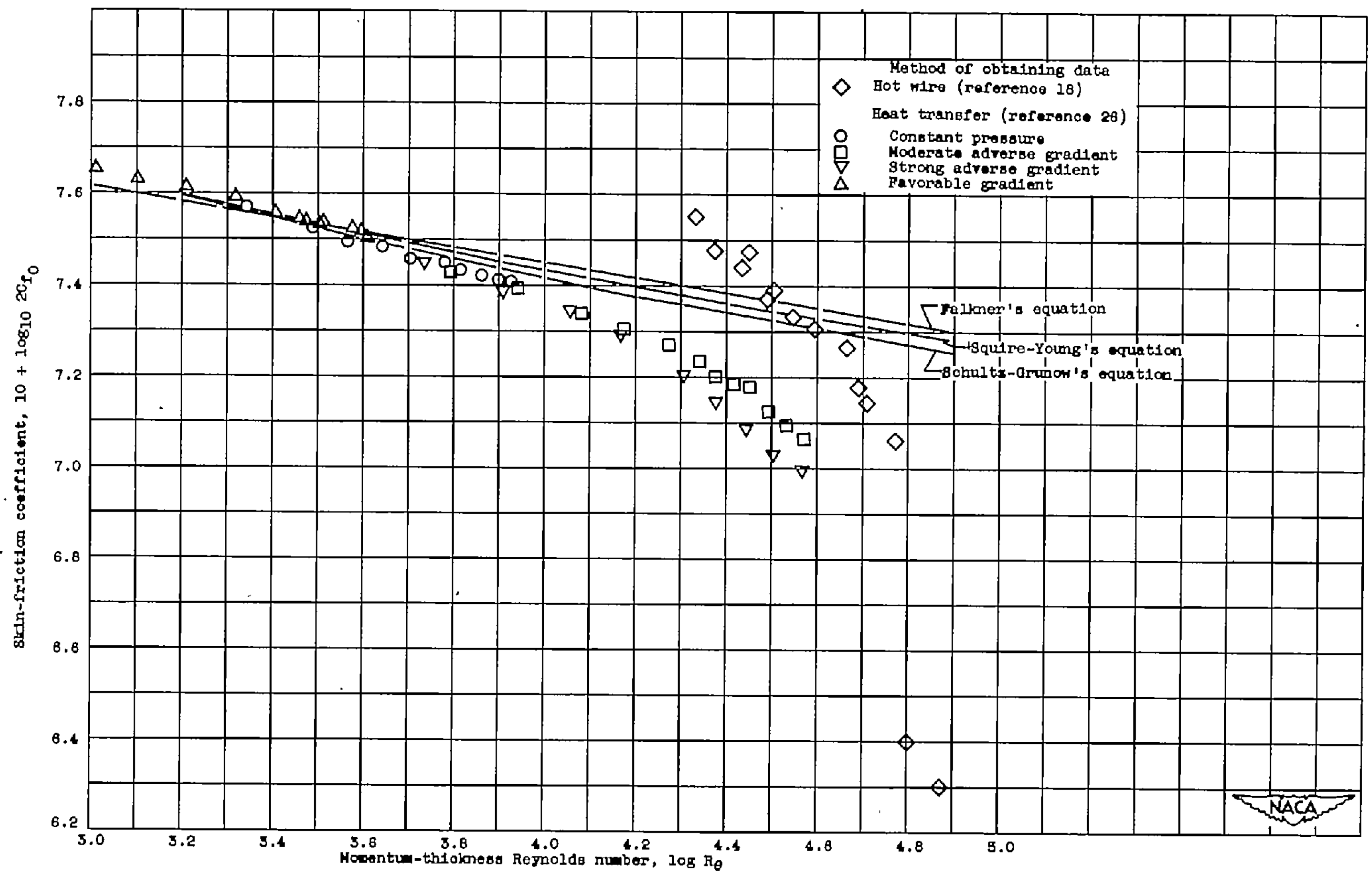
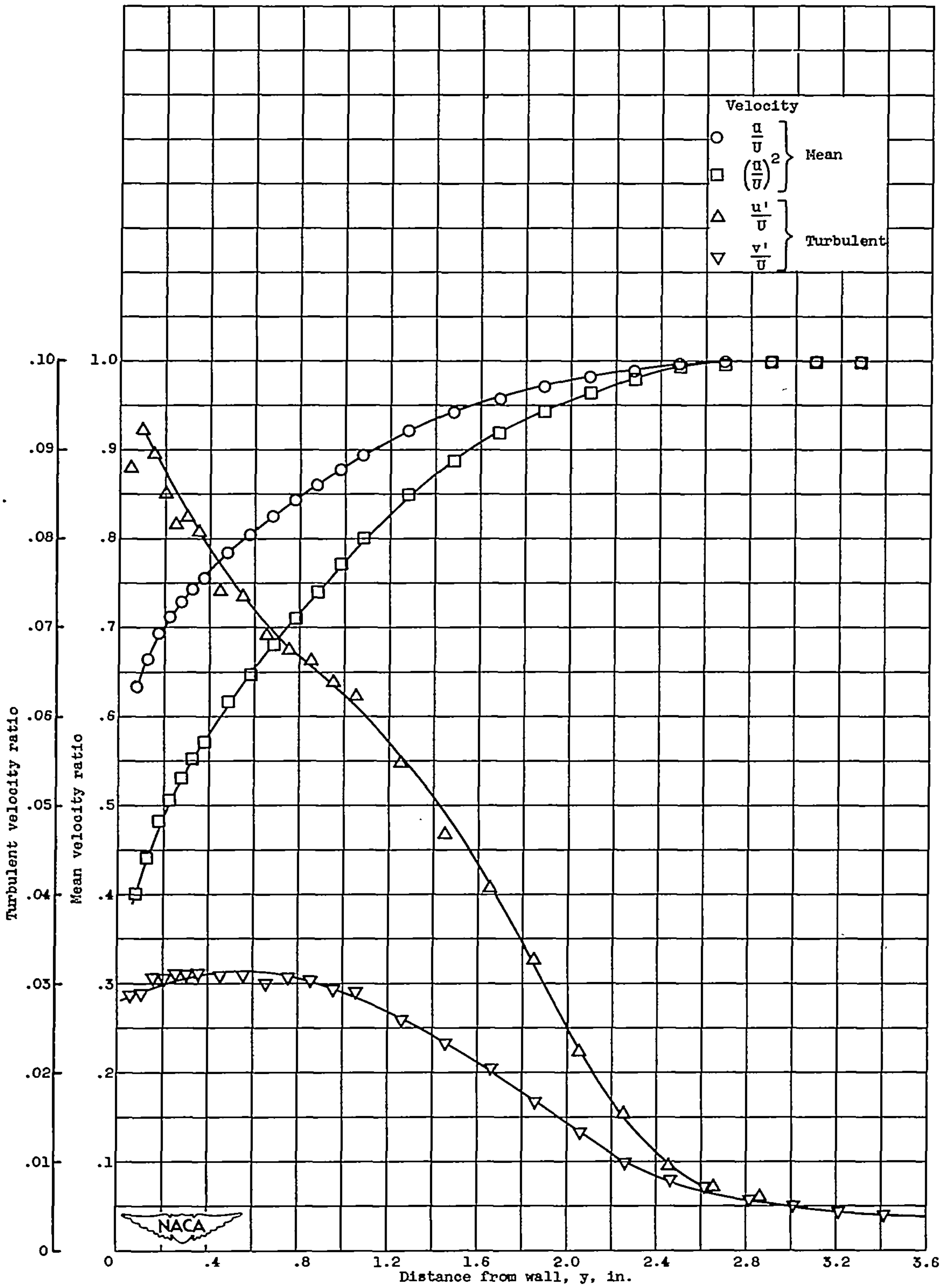
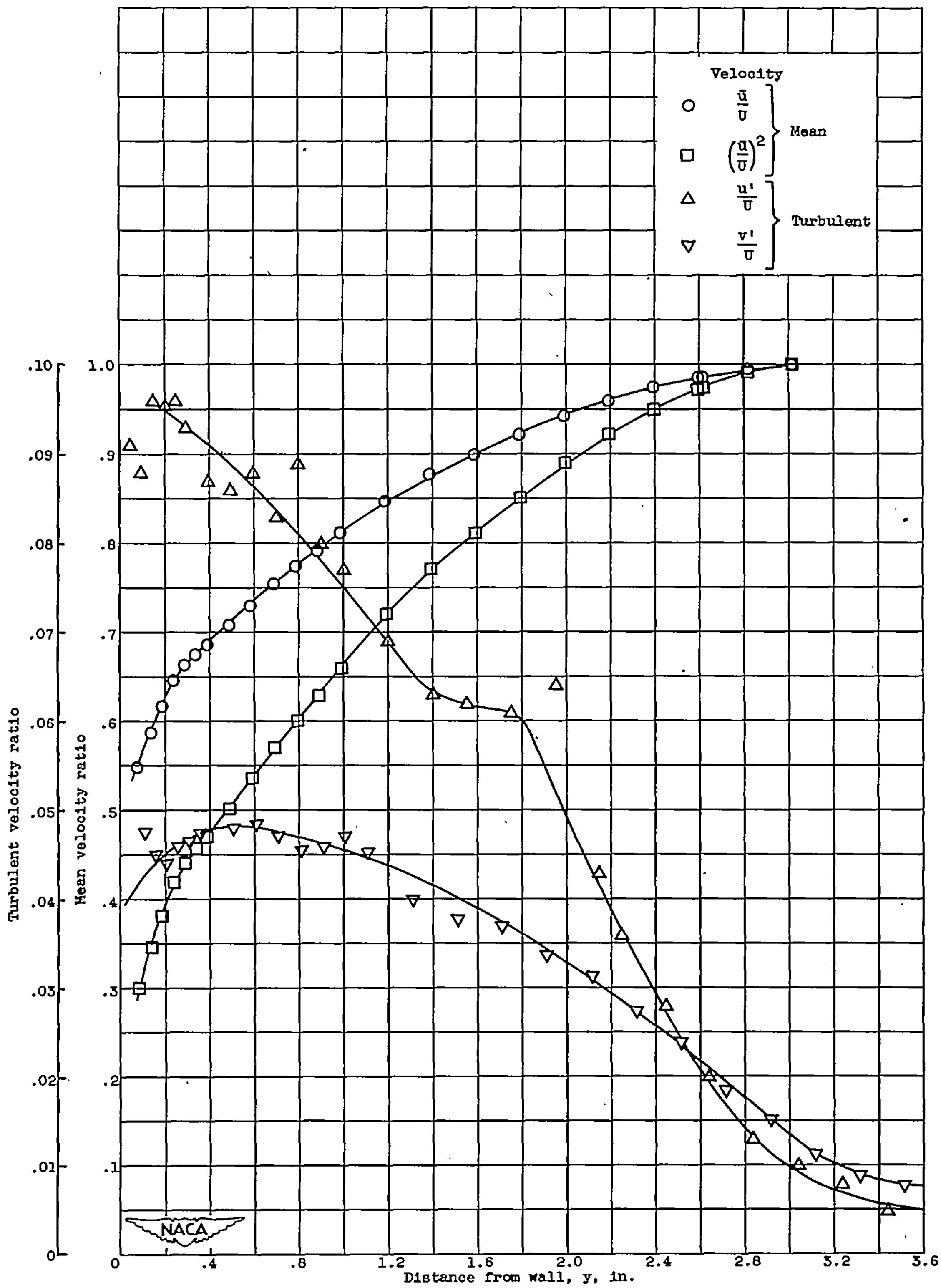


Figure 10. - Skin-friction coefficient under adverse pressure gradient. Data from references 13 and 20.



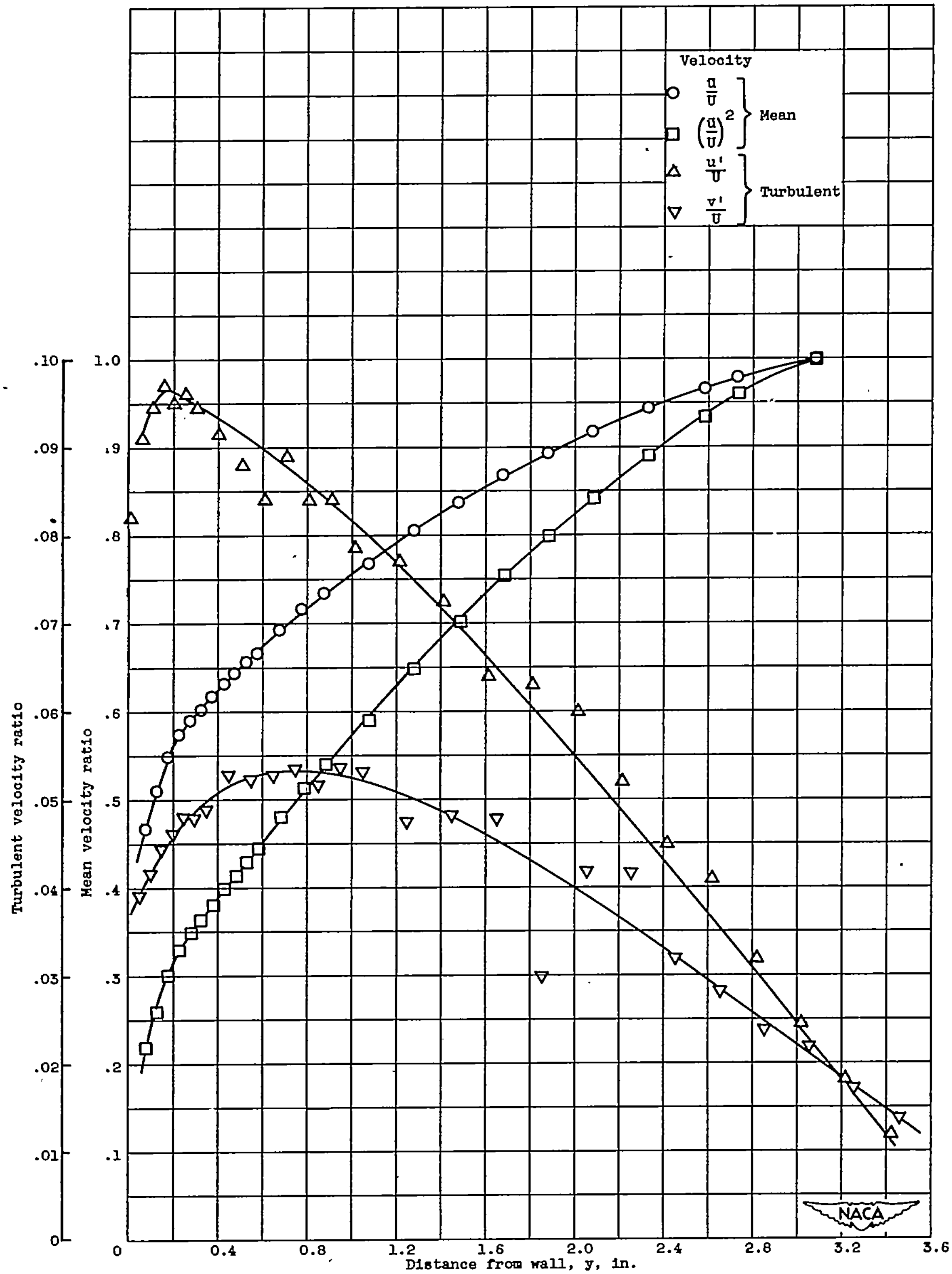
(a) Chordwise distance x, 17.5 feet.

Figure 11. - Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



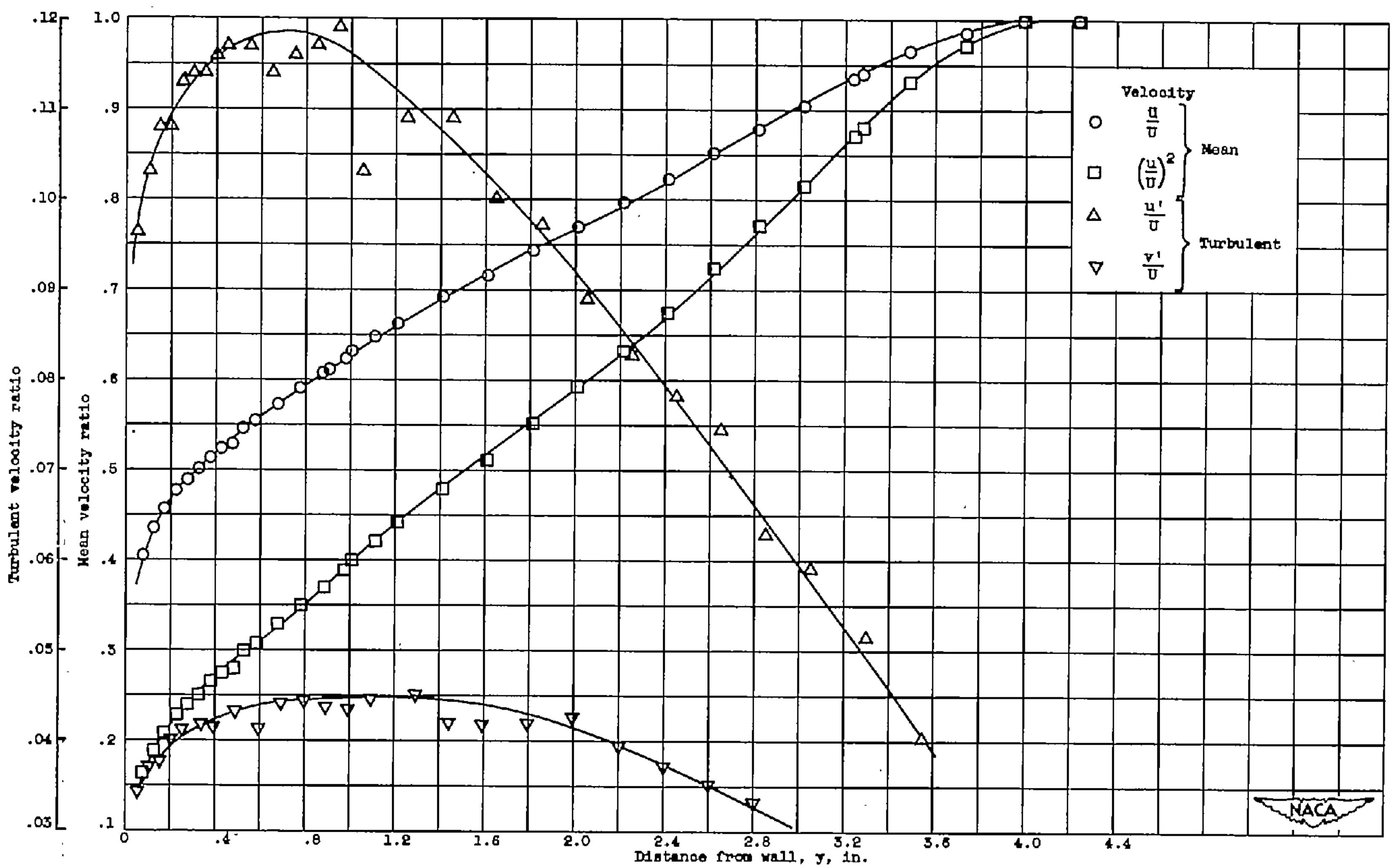
(b) Chordwise distance x, 20.0 feet.

Figure 11. - Continued. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



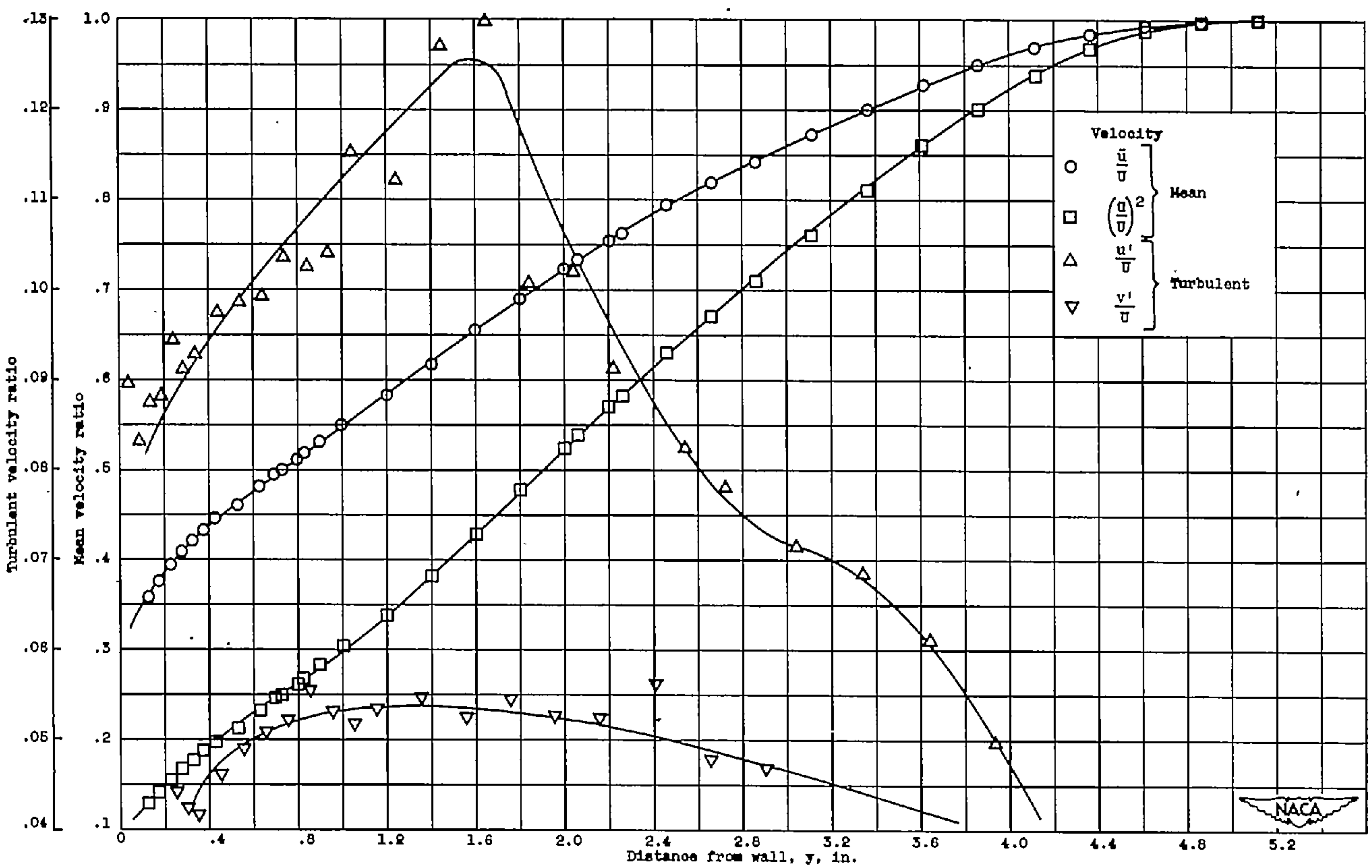
(c) Chordwise distance x, 21.0 feet.

Figure 11. - Continued. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



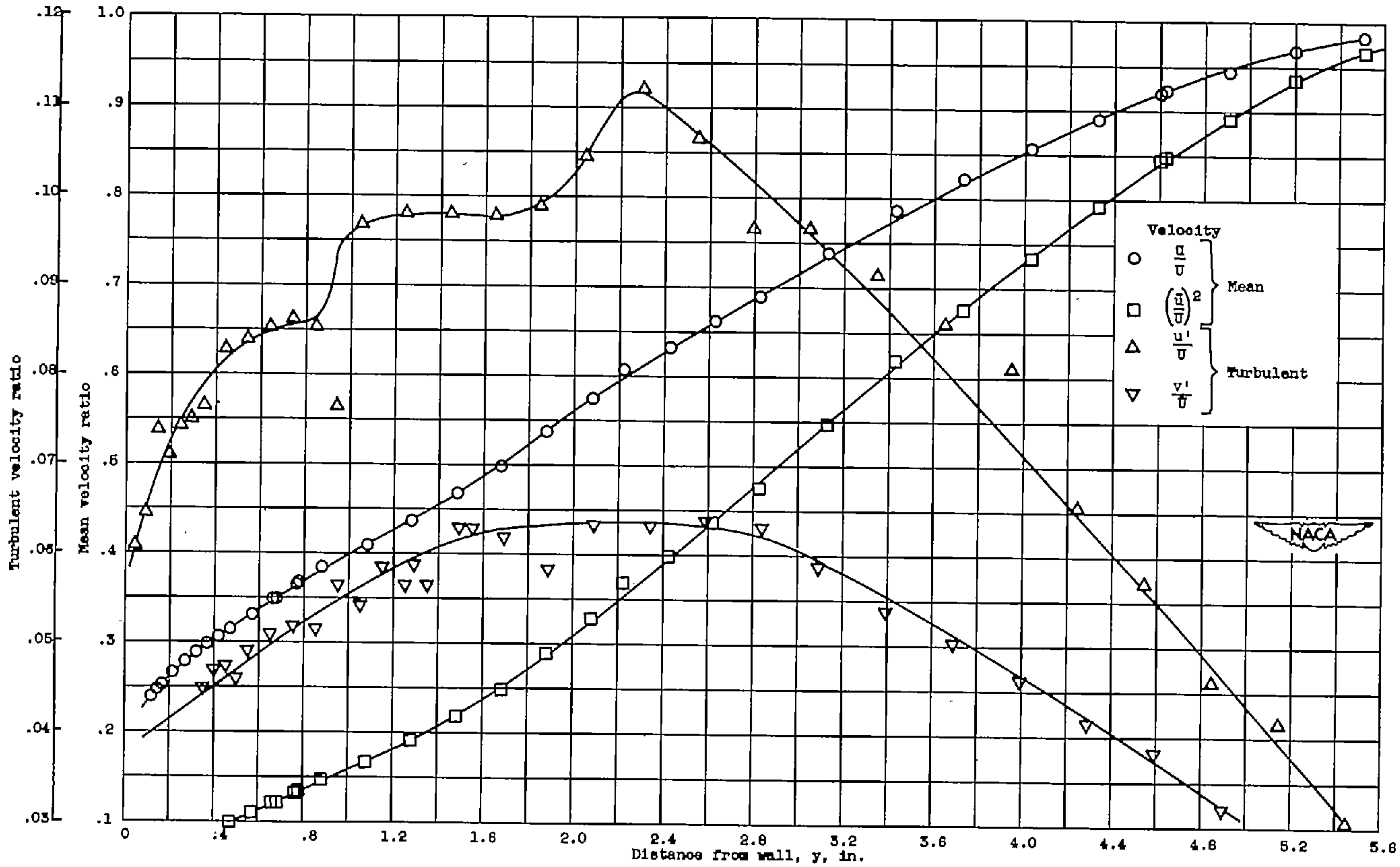
(d) Chordwise distance x, 22.5 feet.

Figure 11. - Continued. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



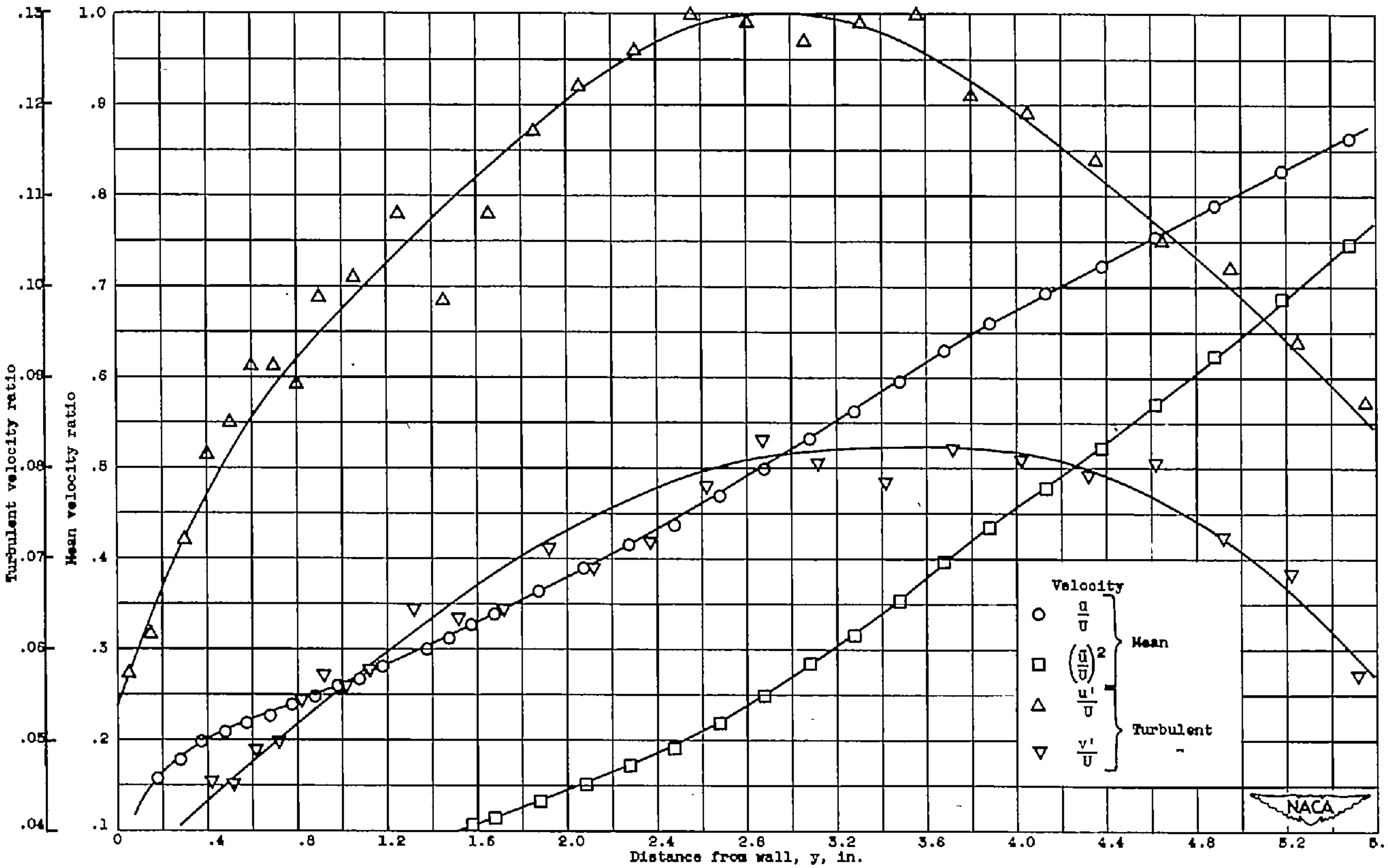
(e) Chordwise distance x, 23.5 feet.

Figure 11. - Continued. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



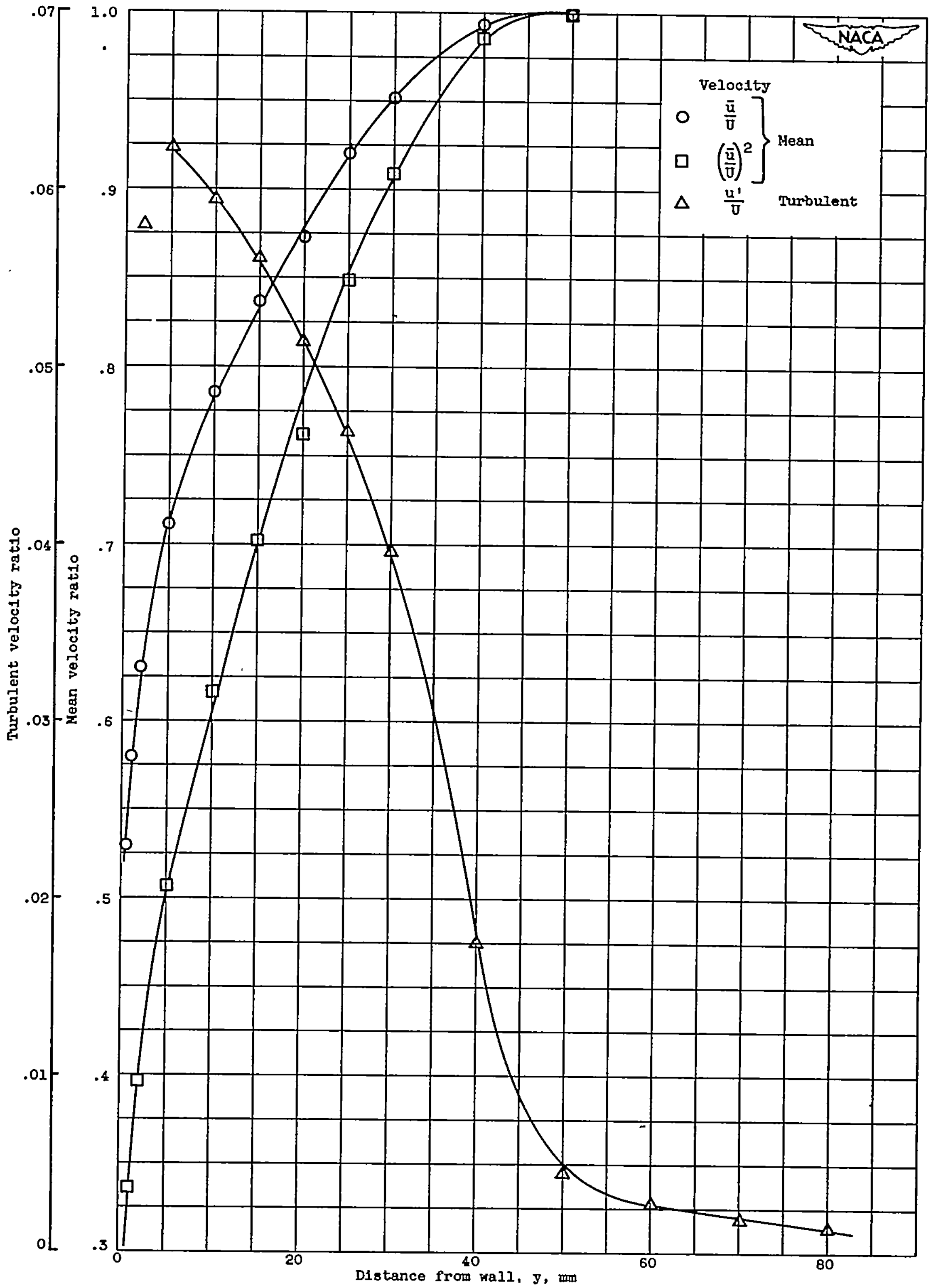
(f) Chordwise distance x, 24.5 feet.

Figure 11. - Continued. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



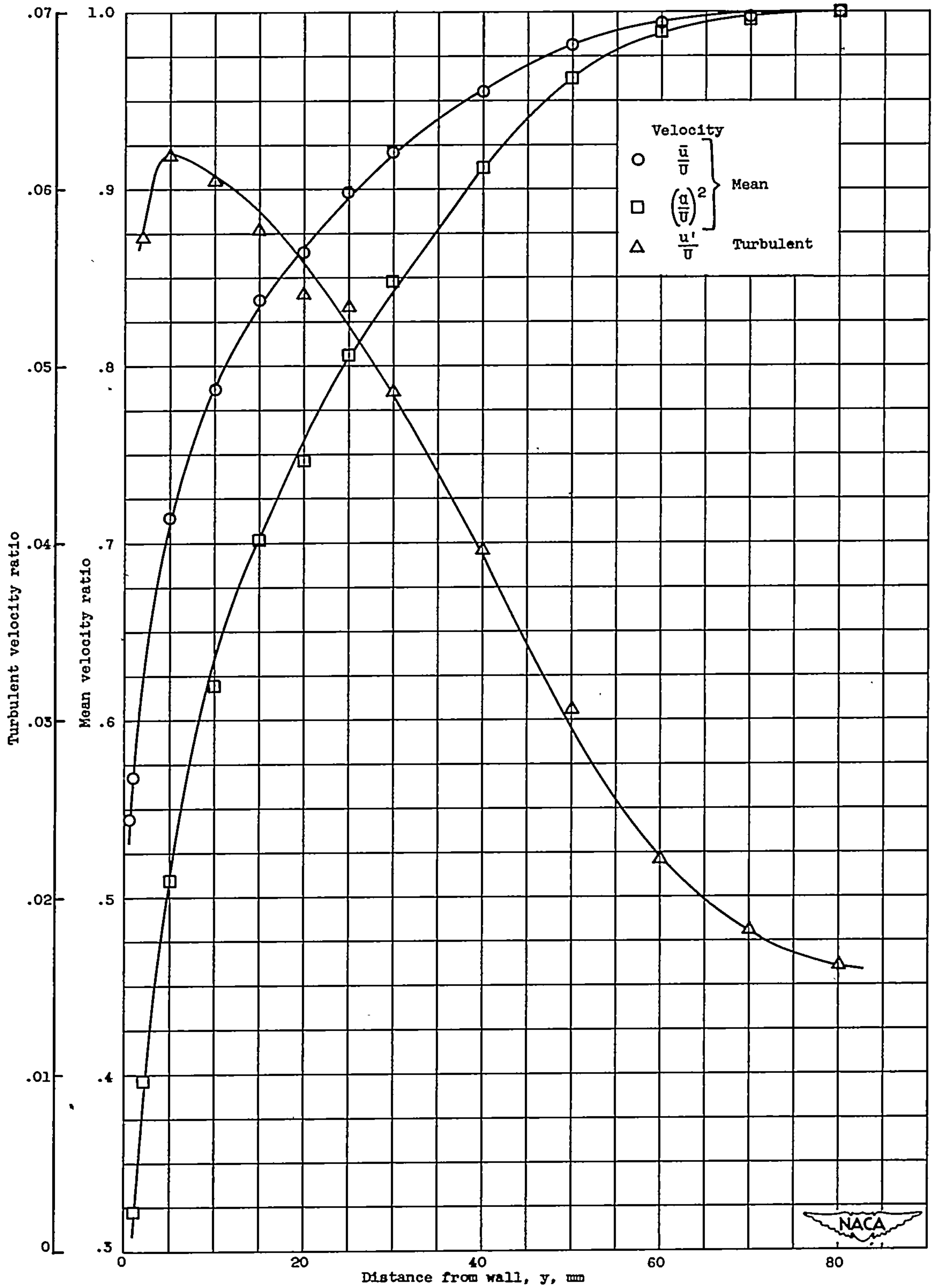
(g) Chordwise distance x, 25.4 feet.

Figure 11. - Concluded. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 18.



(a) Without turbulence grid.

Figure 12. - Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 20.



(b) With turbulence grid.

Figure 12. - Concluded. Mean and turbulent velocity profiles for computation of the laminar and turbulent momentum thicknesses based on data from reference 20.

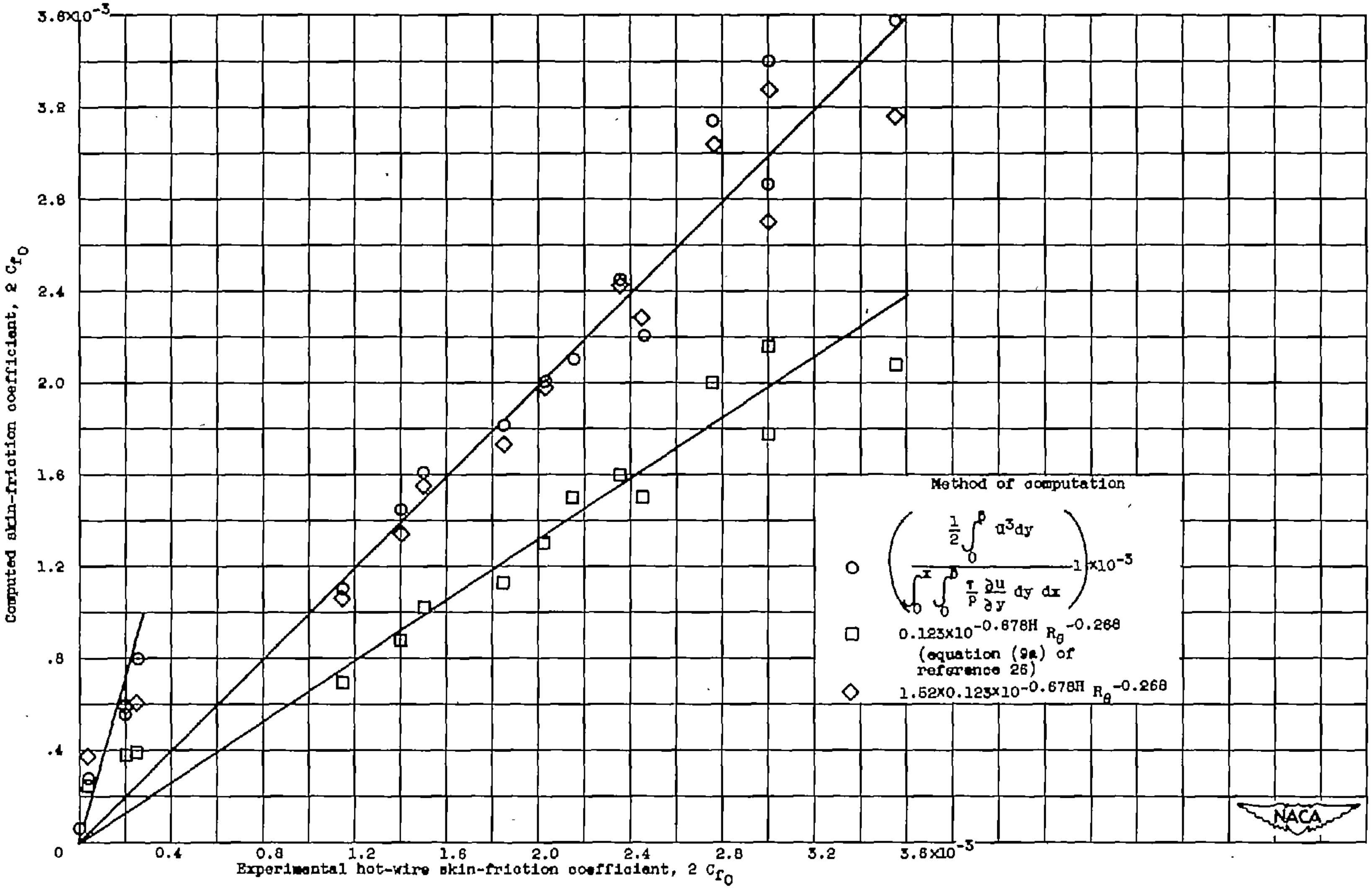


Figure 13. - Relation between experimental hot-wire skin-friction data of reference 18 and values computed by skin-friction formulas.

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